Chapter 2
Finite Field and Linear Block Codes

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References:

Lin, S. and Costello Jr. D.J. , Error Control Coding , Pearson Prentice Hall , 2004.
Castineira, J., and Farrel, P.G. , Essential of Error-Control Coding , Wiley, 2006

2.1 Finite Fields

- **A field with only a finite number of elements is called a finite field. Finite fields are also known as Galois fields after their inventor.**
- **Most of the popular linear block codes , such as Hamming codes. BCH codes and Reed-Solomin codes, are constructed over the finite fields.**
- **For any positive integer $m \geqq 1$, there exists a Galois field of 2^m elements , denoted GF(2^m) . That is an extension field of GF(2) which is the binary field.**
- **Construction of GF(2^m)**

  (1) Begin with a primitive (irreducible) polynomial p(x) of degree m with coefficients from the binary field GF(2).

  (2) Let α be the root of $p(x)$, i.e. $p(\alpha) = 0$

  (3) Starting from GF(2) = { $0,1$) and α , we define a multiplication operator " . " to introduce a sequence of power of 2 as follows :

$0 \cdot 0 = 0$; $0 \cdot 1 = 1 \cdot 0 = 0$;

$1 \cdot 1 = 1$,

$0 \cdot \alpha = \alpha \cdot 0 = 0$;

$1 \cdot \alpha = \alpha \cdot 1 = \alpha$;

$\alpha^2 = \alpha \cdot \alpha$

$\alpha^3 = \alpha \cdot \alpha \cdot \alpha$

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$\alpha^j = \alpha \cdot \alpha^{j-1}$; $\alpha^i \cdot \alpha^j = \alpha^{i+j}$

We now have the following set of elements ,

F = { $0,1$, α ,α^2 … }

which is closed under multiplication "·" .

- Since α is a root of p(x) and p(x) divides $x^{2^m-1}+1$, α must also be a root of $x^{2^m-1}+1$. Hence $\alpha^{2^m-1}+1=0$

  This implies that $\alpha^{2^m-1}=1$

  As a result, F is finite and consists of following elements

  $$F=\{0,1,\alpha,\alpha^2,\ldots,\alpha^{2^m-2}\}$$

- Let $\alpha^0=1$. Multiplication is carried out as follows.

  For $0\leqq i$, $j\leqq 2^m-1$,

  $$\alpha^i \bullet \alpha^j = \alpha^{i+j}=\alpha^k$$

  where k is the remainder resulting from dividing $i+j$ by 2^m-1.

  Since $\alpha^i \bullet \alpha^{2^m-1-i}=\alpha^{2^m-1-i}$,

  $\alpha^{2\ -1-I}$ is called the multiplicative inverse of α^i and vise versa.

  Also, $\alpha^{2^m-1-i}=\alpha^{2^m-1}\bullet\alpha^{-i}=\alpha^{-I}$

  we can use α^{-i} to denote the multiplicative inverse of α^i. 4

- **Next , we define " division " operator as follows :**

  $\alpha^i \div \alpha^j = \alpha^i \cdot \alpha^{-j} = \alpha^{i-j}$

- **The ' addition " operator on F is defined as follows .**

  For $0 \leqq i \leqq 2^m - 2$, dividing x^i by p(x) yields

  $x^i = a(x)\, p(x) + b(x)$

  where b (x) is the remainder and

  $b(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_{m-1} x^{m-1}$

  Replacing *x* by α , we have

  $\alpha^i = a(\alpha)\, p(\alpha) + b(\alpha) = b_0 + b_1 \alpha + b_2 \alpha^2 + \ldots + b_{m-1} \alpha^{m-1}$

  Therefore , each nonzero element in F can be expressed as a polynomial of α with degree *m-1* or less.

- **The "addition" of α^i and α^j is defined as**

  $\alpha^i + \alpha^j = (b_0 + d_0) + (b_1 + d_1)\alpha + \ldots + (b_{m-1} + b_{m-1})\, x^{m-1}$

  where $\alpha^j = d_0 + d_1 \alpha + d_2 \alpha^2 + \ldots + d_{m-1} \alpha^{m-1} = \alpha^k$

- **Clearly , $\alpha^j + \alpha^j = 0$**

  Thus , " subtraction " is defined as follows.

  $$\alpha^i - \alpha^j = \alpha^i + \alpha^j$$

  Hence , subtraction is the same as addition

- **We conclude that F = { 0,1 , α ,α² … } together with the multiplication and addition operators for a field of 2^m elements .**

- **There are three forms to represent the elements in GF (2^m)**

  (1) Power form (easier to perform multiplication)

  $$F = \{ 0,1 , \alpha ,\alpha^2 \ldots \}$$

  (2) Polynomial form

  $$\alpha^i = b_0 + b_1 \alpha + b_2 \alpha^2 + \ldots + b_{m-1} \alpha^{m-1}$$

  (3) Vector form (easier to perform addition)

  $$\alpha^i = (b_0 , b_1 , b_2 , \ldots, b_{m-1})$$

Table B.4 The Galois field GF(2^4) generated by $p_i(X) = 1 + X + X^4$

Exp. representation	Polynomial representation				Vector representation
0	0				0 0 0 0
1	1				1 0 0 0
α		α			0 1 0 0
α^2			α^2		0 0 1 0
α^3				α^3	0 0 0 1
α^4	1	$+\alpha$			1 1 0 0
α^5		α	$+\alpha^2$		0 1 1 0
α^6			$+\alpha^2$	$+\alpha^3$	0 0 1 1
α^7	1	$+\alpha$		$+\alpha^3$	1 1 0 1
α^8	1		$+\alpha^2$		1 0 1 0
α^9		α		$+\alpha^3$	0 1 0 1
α^{10}	1	$+\alpha$	$+\alpha^2$		1 1 1 0
α^{11}		α	$+\alpha^2$	$+\alpha^3$	0 1 1 1
α^{12}	1	$+\alpha$	$+\alpha^2$	$+\alpha^3$	1 1 1 1
α^{13}	1		$+\alpha^2$	$+\alpha^3$	1 0 1 1
α^{14}	1			$+\alpha^3$	1 0 0 1

2.2 Primitive Polynomials and Minimal Polynomials

- **A irreducible polynomial *p*(x) of degree *m* is said to be <u>primitive</u> if the smallest positive integer *n* for which p(x) divides $x^n + 1$ is $n = 2^m - 1$.**
- **For example, $1 + x + x^4$ is a primitive polynomial.**
  The smallest positive integer *n* for which $1 + x + x^4$ divides $x^n + 1$ is $n = 15$.
- **For any positive integer *m*, there exists a primitive polynomial of degree *m* .**
- **Example**

M	Primitive Polynomial *p(x)*
2	$1 + x + x^2$
3	$1 + x + x^3$
4	$1 + x + x^4$
5	$1 + x^2 + x^5$
6	$1 + x + x^6$
7	$1 + x^3 + x^7$

Consider the Galois field GF (2^m) generated by the primitive polynomial

$$p(x) = p_0 + p_1 x + p_2 x^2 + \dots + p_{m-1} x^{m-1} + x^m$$

The element α, which is a root of p(x), whose powers generate all the non-zero elements of GF (2^m) is called a primitive element of GF (2^m). Usually, there may be more than one primitive elements in a finite field GF (2^m).

For example, α^4 and α^7 are also primitive elements of GF (2^m).

- **Let. β be a non-zero element of GF (2^m).**

  Consider the powers of β :

  β, β^2, β^4 β^8, ..., β^{2^i} ,...

  If e is the smallest nonnegative integer for which $\beta^{2^e} = \beta$, then the integer " e " is called the exponent of β

- The minimal polynomial of the element β is defined as

  $$\phi(x) = (x + \beta)(x + \beta^2)(x + \beta^4)\dots(x + \beta^{2^{e-1}})$$

- **Let f(x) be a polynomial defined over GF GF(2^m) . If an element β of GF(2^m) is a root of the polynomial f(x) , then for**

  any positive integer $\lambda \geqq 0$, β^{2^λ} is also a root of that polynomial .

  The elements β^{2^λ} are called conjugates of β .

Theorem 2.1:

If an element β of GF(2^m) is a root of the polynomial f(x) , its conjugates are also elements of the same field and roots of the same polynomial.

- **Theorem 2.2 :**

  The minimal polynomial $\psi(x)$ of the element β of the Galois field GF (2^m) is a factor of $x^{2^m} + x$

- **Example :**

  The following table lists the minimal polynomials of all elements of the Galois field GF (2^4) generated by $p(x) = 1+x + x^4$.

Conjugate roots	Minimal polynomials
0	x
1	$1 + x$
$\alpha, \alpha^2, \alpha^4, \alpha^8$	$1 + x + x^4$
$\alpha^3, \alpha^6, \alpha^9, \alpha^{12}$	$1 + x + x^2 + x^3 + x^4$
α^5, α^{10}	$1 + x + x^2$
$\alpha^7, \alpha^{11}, \alpha^{13}, \alpha^{14}$	$1 + x^3 + x^4$

2.3 Linear Block Codes

- **Let the message m = ($m_0, m_1, \ldots, m_{k-1}$) be an arbitrary k-tuple from GF (2) . The linear (n, k) code over GF (2) is the set 2^k codewords of row vector form**

  c = ($c_0, c_1, \ldots, c_{n-1}$) , where c_j ε GF (2)

  The generator G of the code is a k x n matrix over GF (2) .

  c = m · G

  The generator matrix can be expressed as

  G = [g_1 g_2 … g_k] ^{T}

  The rows of G are linearly independent since G is assumed to have rank k .
- **For a linear block code, the vector sum of two codewords is a codeword.**

- **The generator matrix of an (n, k) linear systematic code can be expressed as**

  $$\mathbf{G} = [\, \mathbf{I}_k \quad \mathbf{P} \,]$$

  where $\mathbf{I}_k$ is the k x k identity matrix and $\mathbf{P}$ is a k x (n-k) matrix.

- **An (n , k) linear code C can also be specified by an (n-k) x k matrix H denoted as parity –check matrix .**

  Let $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ be an n-tuple , then $\mathbf{c}$ is a codeword

  if and only if $\mathbf{c} \cdot \mathbf{H} = (0 , 0, \ldots , 0)_{n-k} = 0$

  The parity-check matrix can be expressed as

  $$\mathbf{H} = [\, \mathbf{P}^T \quad \mathbf{I}_{n-k} \,]$$

  It is noted that many solutions for H are possible for any given generator matrix G .

Example : Hamming code

- Hamming codes are the first class of binary linear block code discovered by R.W. Hamming in 1950.
- For any positive integer $m \geqq 3$, there exists a Hamming code with the following parameters :

  block code length $n = 2^m - 1$

  message length $k = 2^m - 1 - m$

  minimum Hamming distance $d_{min} = 3$

  error-correction capability $t = 1$.

For a (7 ,4) Hamming code

$$G = \begin{bmatrix} 1000\ 101 \\ 0100\ 111 \\ 1101\ 001 \\ 0001\ 011 \end{bmatrix} \qquad H = \begin{bmatrix} 1110100 \\ 0111010 \\ 0010110 \end{bmatrix}$$

2.4 Cyclic Codes

- **An (n,k) linear code C is called a cyclic code if any cyclic shift of a codeword is another codeword .**

  In polynomial form

  $$c(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$$

  $$c^{(j)}(x) = c_{n-j} + c_{n-j+1} x + c_{n-j+2} x^2 + \ldots + c_{n-j-1} x^{n-1}$$

  Cyclic structure makes the encoding and syndrome computation very easy.

2.4.1 Generator Polynomial

- **Every nonzero code polynomial c(x) in C must have degree at least n-k but not greater than n-1 . There is one and only one nonzero generator polynomial g(x) for a cyclic code.**

- **It can be shown that the generator polynomial g(x) of an (n ,k) cyclic code is always a polynomial factor of the polynomial**

  $x^{n} - 1$, or $x^{n} + 1$.

  $g(x) = 1 + g_1 x + g_2 x^{2} + \dots + g_{n-k-1} x^{n-k-1} + x^{n-k}$

  Since g(x) divides $x^{n} - 1$, it follows that

  $x^{n} - 1 = h(x)\, g(x)$

  where $h(x) = h_0 + h_1 x + h_2 x^{2} + \dots + h_k x^{k}$

  and $h_0 = h_k = 1$

  h(x) is called the parity polynomial of the (n , k) cyclic code.

- **The message polynomial is expressed as**

  $m(x) = m_0 + m_1 x + m_2 x^{2} + \dots + m_{k-1} x^{k-1}$

  Then , the product m(x)g(x) is the polynomial representing the code word polynomial of degree n-1 or less.

In general, $c(x)$ and $c^{(j)}(x)$ are related by the formula

$$c^{(j)}(x) = x^{j}\, c(x) \bmod (x^{n} - 1)$$

We can see that

$$c^{(j)}(x) = x^{j}\, m(x)g(x) \bmod (x^{n} - 1) = m^{j}(x)\, g(x)$$

2.4.2 Encoding of Cyclic Codes

- Consider an (n ,k) cyclic code with generator polynomial $g(x)$
  Suppose $m = (m_0, m_1, \ldots, m_{k-1})$ is the message to be encoded.

$$m(x) = m_0 + m_1 x + m_2 x^{2} + \ldots + m_{k-1} x^{k-1}$$

Multiplying $m(x)$ by x^{n-k} and the dividing by $g(x)$, we obtain

$$x^{n-k}\, m(x) = q(x)\, g(x) + p(x)$$

where $p(x) = p_0 + p_1 x + p_2 x^{2} + \ldots + p_{n-k-1} x^{n-k-1}$

is the remainder .

Then $p(x) + x^{n-k} m(x) = q(x)g(x)$ is a multiple of $g(x)$ and has degree n-1. Hence it is the code polynomial for the message.

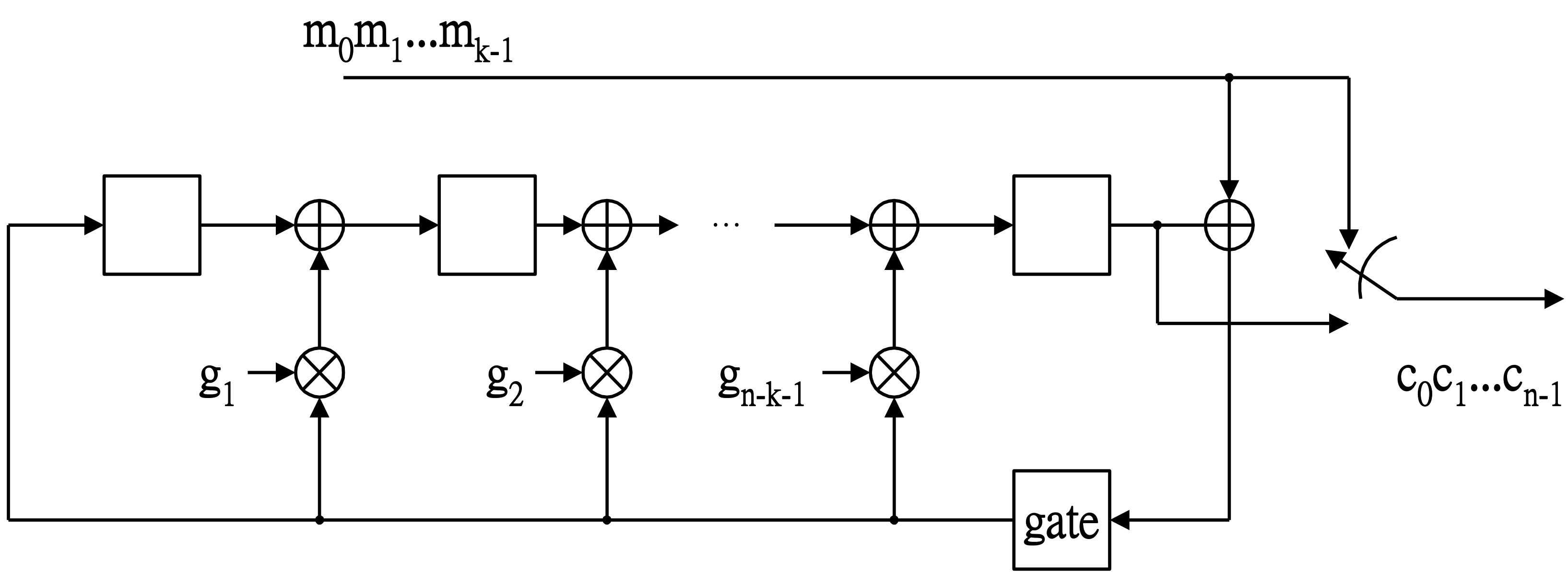
- **Note that**

$$p(x) + x^{n-k}\, m(x)$$
$$= p_0 + p_1 x + p_2 x^{2} + \ldots + p_{n-k-1} x^{n-k-1} +$$
$$m_0 x^{n-k} + m_1 x^{n-k+1} + \ldots + m_{k-1} x^{k-1}$$

The code polynomial is in systematic form where $p(x)$ is the parity-check part .

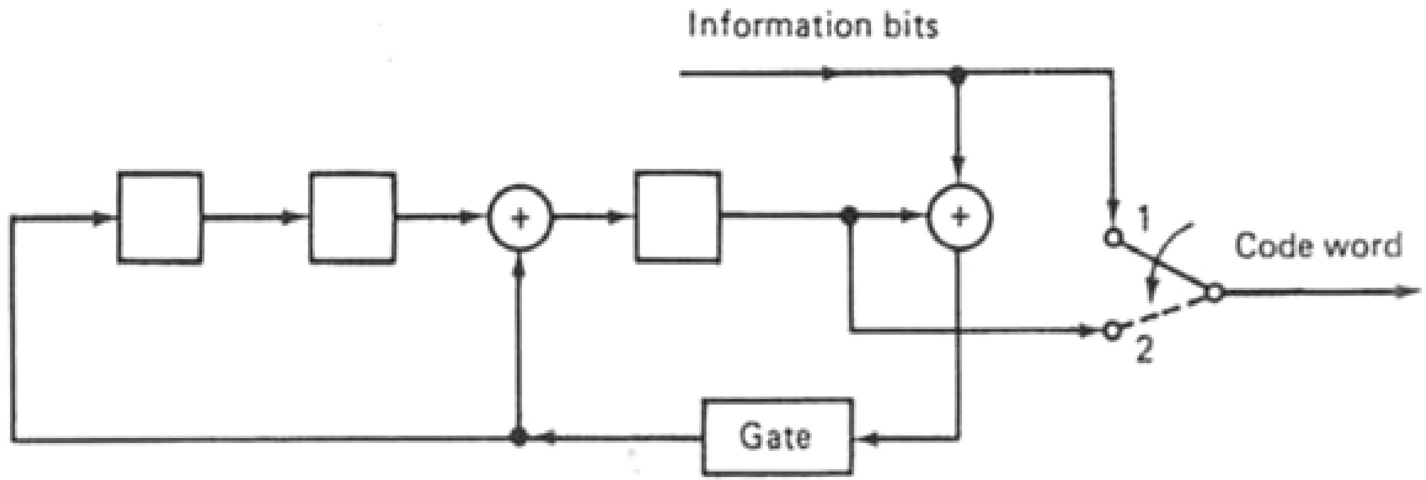
- **The encoding can be implemented by using a division circuit consisting of shift registers and feedback connections based on the generator polynomial g(x) ,as show below Fig.2.1) .**
- **In the figure the right-most symbol is the first symbol to enter the encoder. The gate is turned on until all information digits have been shifted into the circuit.**

Fig.2.1 Encoding circuit based on g(x)



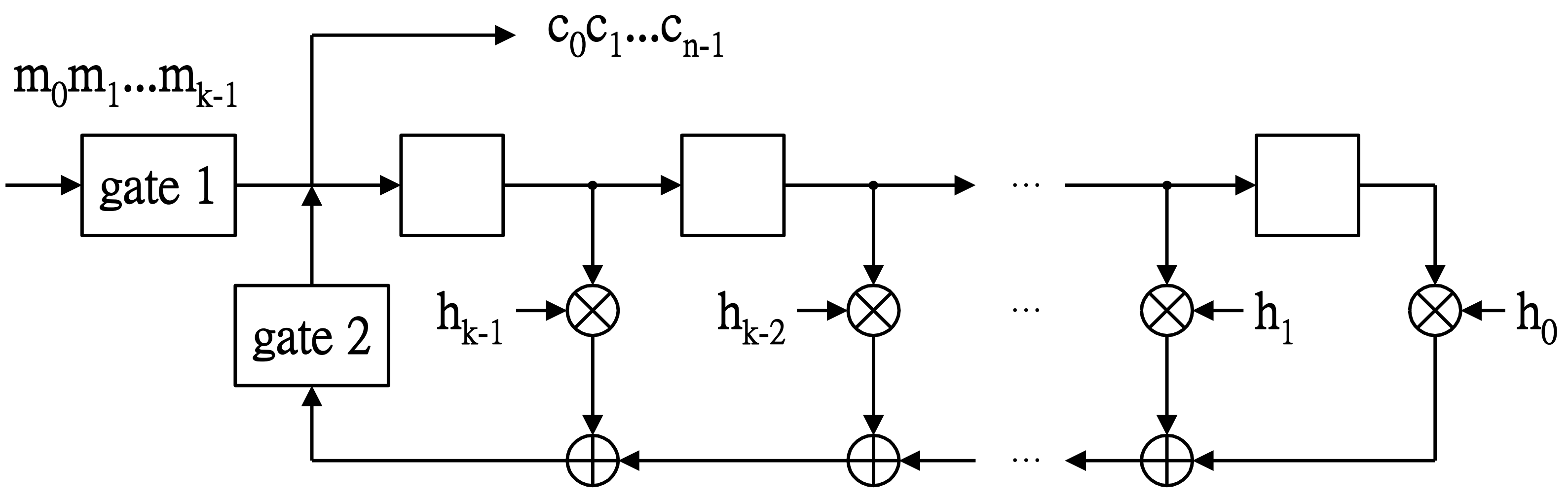
Example : Encoding of cyclic (7,4) Hamming code

$g(x) = 1+x^{2}+x^{3}$, message bits m = (1001)



Shift no. i	Gate	After ith shift Register contents	Output
0	On	0 0 0	1
1	On	1 0 1	0 1
2	On	1 1 1	0 0 1
3	On	1 1 0	1 0 0 1
4	Off	1 1 0	0 1 0 0 1
5	Off	0 1 1	1 0 1 0 0 1
6	Off	0 0 1	1 1 0 1 0 0 1

- **It can be shown that cyclic codes can also be generated by using the parity polynomial h(x) , where h(x) =** $h_0 + h_1 x + h_2 x^2 + \ldots + h_k x^k$ **.**
- **The k-stage shifter-register encoder based on h(x) is shown in Fig.2.2 .**



2.5 Syndrome Computation

- **Let c(x)and r(x) be the transmitted code polynomial and received polynomial, respectively.**

  Dividing $r(x)$ by the generator polynomial g (x) , we have

  $$r(x) = q(x)\, g(x) + s(x)$$

  where s(x) is the remainder and

  $$s(x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_{n-k-1} x^{n-k-1}$$

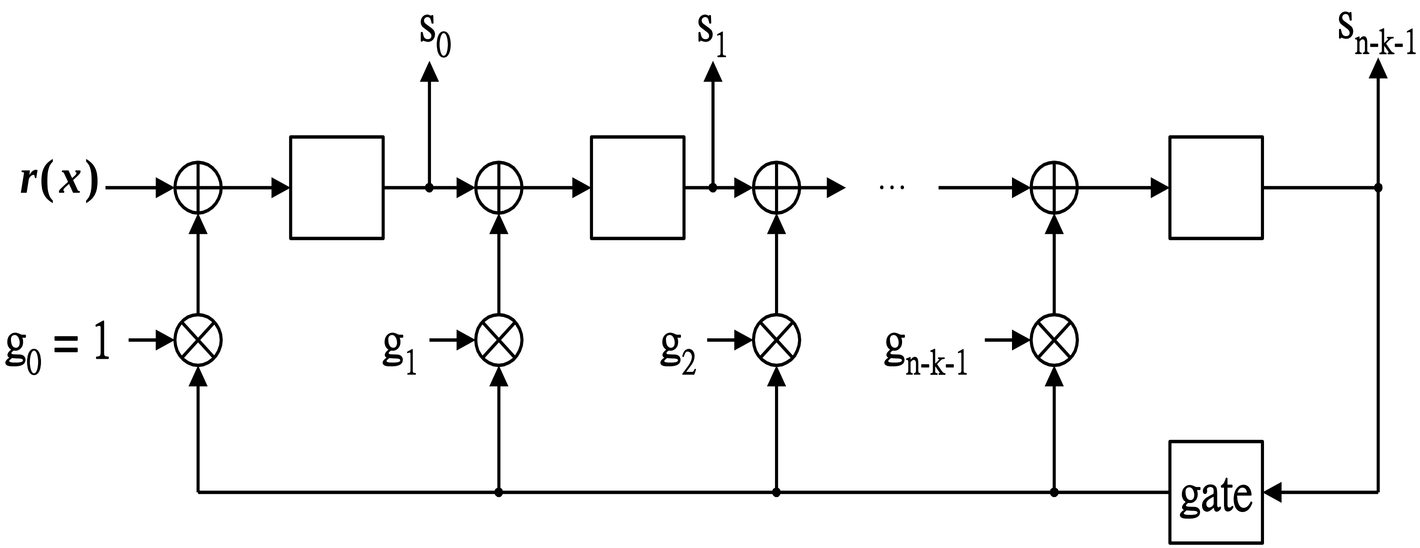
  Then s(x) is the syndrome polynomial of r(x).

  The received polynomial r(x) is a code polynomial if and only if $s(x) = 0$.

- **Syndrome computation can be done by a division circuit shown in Fig.2.3 .**

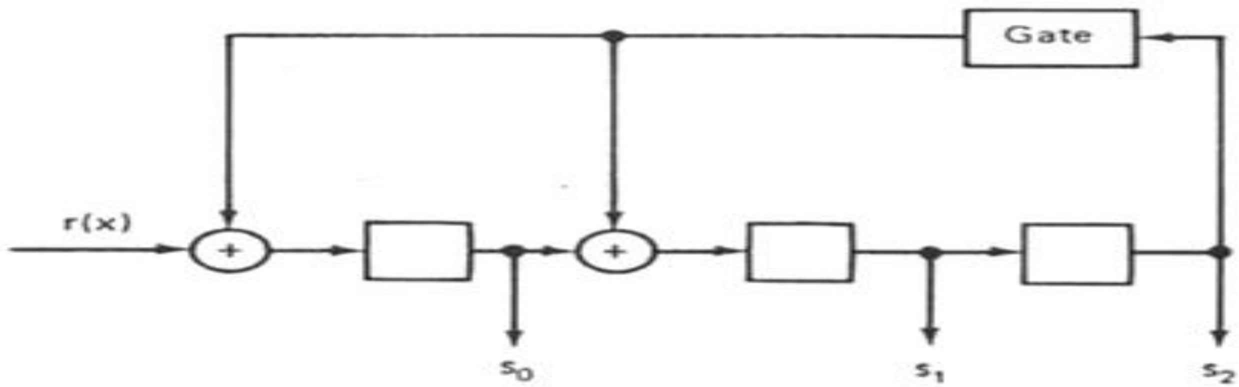
  As soon as the entire r(x) has been shifted into the register , the contents in the register form the $s(x)$.

Fig. 2.3 Syndrome Computation Circuit



Example : Syndrome circuit for a (7,4) cyclic code with $g(x) = 1+ x +x^3$

Received sequence r = (1001000)



Shift no.	Input	Register contents s_0, s_1, s_2
0	—	0 0 0
1	0	0 0 0
2	0	0 0 0
3	0	0 0 0
4	1	1 0 0
5	0	0 1 0
6	0	0 0 1
7	1	0 1 0
8	0	0 0 1
9	0	1 1 0
10	0	0 1 1
11	0	1 1 1
12	0	1 0 1

- **Since** $r(x) = c(x) + e(x)$

  and also $r(x) = q(x)\, g(x) + s(x)$

  we have $e(x) = r(x) + c(x)$

$$= q(x)g(x) + s(x) + c(x)$$

$$= q(x)\, g(x) + s(x) + m(x)\, g(x)$$

$$= [\, q(x) + m(x)\,]\, g(x) + s(x)$$

  or $s(x) = e(x) \bmod g(x)$

  Hence the syndrome polynomial $s(x)$ **is also the remainder that results from dividing** $e(x)$ **by** $g(x)$ **.**

Table 2.1

Galois field GF(2^5) constructed by using the primitive polynomial

$p(x) = 1 + x^2 + x^5$

Field element (polynomial notation)	5-tuple representation
0	0 0 0 0 0
1	1 0 0 0 0
α^1	0 1 0 0 0
α^2	0 0 1 0 0
α^3	0 0 0 1 0
α^4	0 0 0 0 1
$\alpha^5 = 1 + \alpha^2$	1 0 1 0 0
$\alpha^6 = \alpha + \alpha^3$	0 1 0 1 0
$\alpha^7 = \alpha^2 + \alpha^4$	0 0 1 0 1
$\alpha^8 = 1 + \alpha^2 + \alpha^3$	1 0 1 1 0
$\alpha^9 = \alpha + \alpha^3 + \alpha^4$	0 1 0 1 1
$\alpha^{10} = 1 + \alpha^4$	1 0 0 0 1
$\alpha^{11} = 1 + \alpha + \alpha^2$	1 1 1 0 0
$\alpha^{12} = \alpha + \alpha^2 + \alpha^3$	0 1 1 1 0
$\alpha^{13} = \alpha^2 + \alpha^3 + \alpha^4$	0 0 1 1 1
$\alpha^{14} = 1 + \alpha^2 + \alpha^3 + \alpha^4$	1 0 1 1 1
$\alpha^{15} = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$	1 1 1 1 1
$\alpha^{16} = 1 + \alpha + \alpha^3 + \alpha^4$	1 1 0 1 1
$\alpha^{17} = 1 + \alpha + \alpha^4$	1 1 0 0 1
$\alpha^{18} = 1 + \alpha$	1 1 0 0 0
$\alpha^{19} = \alpha + \alpha^2$	0 1 1 0 0
$\alpha^{20} = \alpha^2 + \alpha^3$	0 0 1 1 0
$\alpha^{21} = \alpha^3 + \alpha^4$	0 0 0 1 1
$\alpha^{22} = 1 + \alpha^2 + \alpha^4$	1 0 1 0 1
$\alpha^{23} = 1 + \alpha + \alpha^2 + \alpha^3$	1 1 1 1 0
$\alpha^{24} = \alpha + \alpha^2 + \alpha^3 + \alpha^4$	0 1 1 1 1
$\alpha^{25} = 1 + \alpha^3 + \alpha^4$	1 0 0 1 1
$\alpha^{26} = 1 + \alpha + \alpha^2 + \alpha^4$	1 1 1 0 1
$\alpha^{27} = 1 + \alpha + \alpha^3$	1 1 0 1 0
$\alpha^{28} = \alpha + \alpha^2 + \alpha^4$	0 1 1 0 1
$\alpha^{29} = 1 + \alpha^3$	1 0 0 1 0
$\alpha^{30} = \alpha + \alpha^4$	0 1 0 0 1

Table 2.2 Minimal polynomials of the elements in $GF(2^6)$

Elements	Minimal polynomials
$\alpha, \alpha^2, \alpha^4, \alpha^{16}, \alpha^{32}$	$1 + X + X^6$
$\alpha^3, \alpha^6, \alpha^{12}\alpha^{24}, \alpha^{48}\alpha^{33}$	$1 + X + X^2 + X^4 + X^6$
$\alpha^5, \alpha^{10}, \alpha^{20}, \alpha^{40}, \alpha^{17}, \alpha^{34}$	$1 + X + X^2 + X^5 + X^6$
$\alpha^7, \alpha^{14}, \alpha^{28}, \alpha^{56}, \alpha^{49}, \alpha^{35}$	$1 + X^3 + X^6$
$\alpha^9, \alpha^{18}, \alpha^{36}$	$1 + X^2 + X^3$
$\alpha^{11}, \alpha^{22}, \alpha^{44}, \alpha^{25}, \alpha^{50}, \alpha^{37}$	$1 + X^2 + X^3 + X^5 + X^6$
$\alpha^{13}, \alpha^{26}, \alpha^{52}, \alpha^{41}, \alpha^{19}, \alpha^{38}$	$1 + X + X^3 + X^4 + X^6$
$\alpha^{15}, \alpha^{30}, \alpha^{60}, \alpha^{57}, \alpha^{51}, \alpha^{39}$	$1 + X^2 + X^4 + X^5 + X^6$
α^{21}, α^{42}	$1 + X + X^2$
$\alpha^{23}, \alpha^{46}, \alpha^{29}, \alpha^{58}, \alpha^{53}, \alpha^{43}$	$1 + X + X^4 + X^5 + X^6$
$\alpha^{27}, \alpha^{54}, \alpha^{45}$	$1 + X + X^3$
$\alpha^{31}, \alpha^{62}, \alpha^{61}, \alpha^{59}, \alpha^{55}, \alpha^{47}$	$1 + X^5 + X^6$

Table 2.3

Galois field GF(2^6) constructed by using the primitive polynomial $p(x) = 1 + x + x^6$

0	0						$(0\,0\,0\,0\,0\,0)$
1	1						$(1\,0\,0\,0\,0\,0)$
α		α					$(0\,1\,0\,0\,0\,0)$
α^2			α^2				$(0\,0\,1\,0\,0\,0)$
α^3				α^3			$(0\,0\,0\,1\,0\,0)$
α^4					α^4		$(0\,0\,0\,0\,1\,0)$
α^5						α^5	$(0\,0\,0\,0\,0\,1)$
α^6	1	$+\ \alpha$					$(1\,1\,0\,0\,0\,0)$
α^7		α	$+\ \alpha^2$				$(0\,1\,1\,0\,0\,0)$
α^8			α^2	$+\ \alpha^3$			$(0\,0\,1\,1\,0\,0)$
α^9				α^3	$+\ \alpha^4$		$(0\,0\,0\,1\,1\,0)$
α^{10}					α^4	$+\ \alpha^5$	$(0\,0\,0\,0\,1\,1)$
α^{11}	1	$+\ \alpha$				$+\ \alpha^5$	$(1\,1\,0\,0\,0\,1)$
α^{12}	1		$+\ \alpha^2$				$(1\,0\,1\,0\,0\,0)$
α^{13}		α		α^3			$(0\,1\,0\,1\,0\,0)$
α^{14}			α^2		$+\ \alpha^4$		$(0\,0\,1\,0\,1\,0)$
α^{15}				α^3		$+\ \alpha^5$	$(0\,0\,0\,1\,0\,1)$
α^{16}	1	$+\ \alpha$			$+\ \alpha^4$		$(1\,1\,0\,0\,1\,0)$
α^{17}		α	$+\ \alpha^2$			$+\ \alpha^5$	$(0\,1\,1\,0\,0\,1)$
α^{18}	1	$+\ \alpha$	$+\ \alpha^2$	$+\ \alpha^3$			$(1\,1\,1\,1\,0\,0)$
α^{19}		α	$+\ \alpha^2$	$+\ \alpha^3$	$+\ \alpha^4$		$(0\,1\,1\,1\,1\,0)$
α^{20}			α^2	$+\ \alpha^3$	$+\ \alpha^4$	$+\ \alpha^5$	$(0\,0\,1\,1\,1\,1)$

TABLE 6.2: (*continued*)

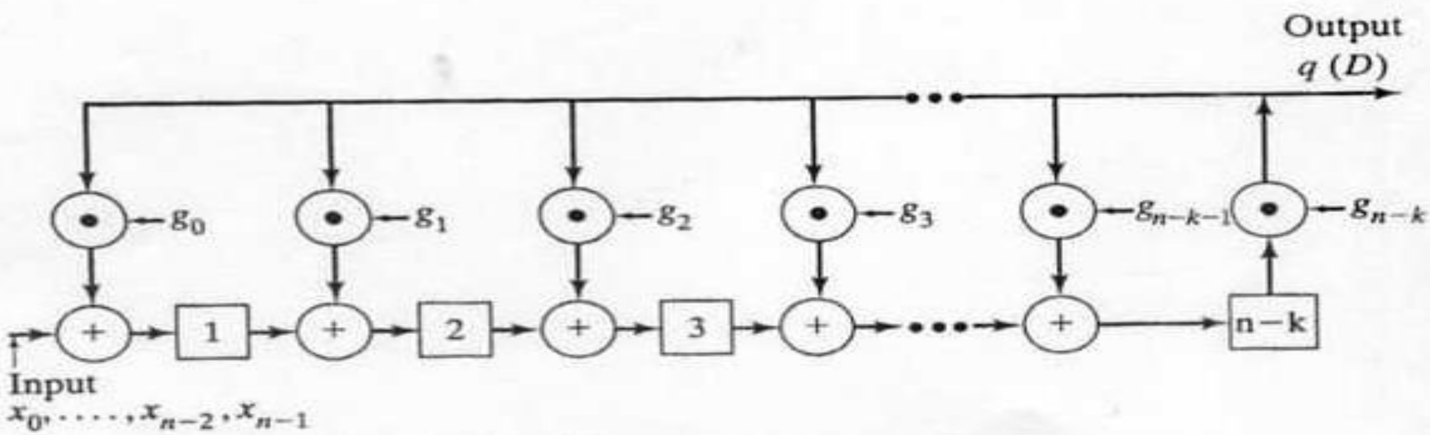
α^{21}	$1 + \alpha + \alpha^3 + \alpha^4 + \alpha^5$	(1 1 0 1 1 1)
α^{22}	$1 + \alpha^2 + \alpha^4 + \alpha^5$	(1 0 1 0 1 1)
α^{23}	$1 + \alpha^3 + \alpha^5$	(1 0 0 1 0 1)
α^{24}	$1 + \alpha^4$	(1 0 0 0 1 0)
α^{25}	$\alpha + \alpha^5$	(0 1 0 0 0 1)
α^{26}	$1 + \alpha + \alpha^2$	(1 1 1 0 0 0)
α^{27}	$\alpha + \alpha^2 + \alpha^3$	(0 1 1 1 0 0)
α^{28}	$\alpha^2 + \alpha^3 + \alpha^4$	(0 0 1 1 1 0)
α^{29}	$\alpha^3 + \alpha^4 + \alpha^5$	(0 0 0 1 1 1)
α^{30}	$1 + \alpha$	(1 1 0 0 1 1)
α^{31}	$1 + \alpha^2 + \alpha^5$	(1 0 1 0 0 1)
α^{32}	$1 + \alpha^3$	(1 0 0 1 0 0)
α^{33}	$\alpha \quad \alpha^4$	(0 1 0 0 1 0)
α^{34}	$\alpha^2 + \alpha^5$	(0 0 1 0 0 1)
α^{35}	$1 + \alpha + \alpha^3$	(1 1 0 1 0 0)
α^{36}	$\alpha + \alpha^2 + \alpha^4$	(0 1 1 0 1 0)
α^{37}	$\alpha^2 \quad \alpha^3 + \alpha^5$	(0 0 1 1 0 1)
α^{38}	$1 + \alpha + \alpha^3 + \alpha^4$	(1 1 0 1 1 0)
α^{39}	$\alpha + \alpha^2 + \alpha^4 + \alpha^5$	(0 1 1 0 1 1)
α^{40}	$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^5$	(1 1 1 1 0 1)
α^{41}	$1 + \alpha^2 + \alpha^3 + \alpha^4$	(1 0 1 1 1 0)
α^{42}	$\alpha + \alpha^3 + \alpha^4 + \alpha^5$	(0 1 0 1 1 1)
α^{43}	$1 + \alpha + \alpha^2 + \alpha^4 + \alpha^5$	(1 1 1 0 1 1)
α^{44}	$1 + \alpha^2 + \alpha^3 + \alpha^5$	(1 0 1 1 0 1)
α^{45}	$1 + \alpha^3 + \alpha^4$	(1 0 0 1 1 0)
α^{46}	$\alpha + \alpha^4 + \alpha^5$	(0 1 0 0 1 1)
α^{47}	$1 + \alpha + \alpha^2 + \alpha^5$	(1 1 1 0 0 1)
α^{48}	$1 + \alpha^2 + \alpha^3$	(1 0 1 1 0 0)
α^{49}	$\alpha + \alpha^3 + \alpha^4$	(0 1 0 1 1 0)
α^{50}	$\alpha^2 + \alpha^4 \quad \alpha^5$	(0 0 1 0 1 1)
α^{51}	$1 + \alpha + \alpha^3 + \alpha^5$	(1 1 0 1 0 1)
α^{52}	$1 + \alpha^2 + \alpha^4$	(1 0 1 0 1 0)
α^{53}	$\alpha + \alpha^3 + \alpha^5$	(0 1 0 1 0 1)
α^{54}	$1 + \alpha + \alpha^2 + \alpha^4$	(1 1 1 0 1 0)
α^{55}	$\alpha + \alpha^2 + \alpha^3 + \alpha^5$	(0 1 1 1 0 1)
α^{56}	$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$	(1 1 1 1 1 0)
α^{57}	$\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5$	(0 1 1 1 1 1)
α^{58}	$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5$	(1 1 1 1 1 1)
α^{59}	$1 + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5$	(1 0 1 1 1 1)
α^{60}	$1 + \alpha^3 + \alpha^4 + \alpha^5$	(1 0 0 1 1 1)
α^{61}	$1 + \alpha^4 + \alpha^5$	(1 0 0 0 1 1)
α^{62}	$1 + \alpha^5$	(1 0 0 0 0 1)

$\boxed{\alpha^{63} = 1}$

Appen.: Division circuit for dividing X(D) by G(D)

$$X(D) = x_0 + x_1 D + x_2 D^2 + \ldots + x_{n-1} D^{n-1}$$

$$G(D) = g_0 + g_1 D + g_2 D^2 + \ldots + g_{n-k} D^{n-k}$$



1) Input high order coefficients first

2) First output is coefficient of D^{n-1} of quotient (always equal to zero but mentioned here to associate outputs with correct power of D in quotient) and is present before first shift register clock pulse

3) First non- zero output occurs after $(n-k)^{\text{th}}$ clock pulse and is coefficient of D^{n-k} in quotient

4) Last term of quotient appears at output after $(n-1)^{\text{th}}$ clock pulse and is coefficient of D^0 in quotient

5) Shift register contains coefficients of remainder $r(D) = r_0 + r_1 D + \ldots r_{n-k-1} D^{n-k-1}$ from left to right after n^{th} clock pulse

Divider circuit using linear feedback shift register structure

$$G(D) = \frac{C(D)}{M(D)} = \frac{a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n}{1 + f_1 D + f_2 D^2 + \cdots + f_n D^n} \tag{28}$$