Contents

2.3 Linear Estimation

- **2.3.1 Signal Estimation Problem**
- **2.3.2 Optimum Estimation of Signals**
- **2.3.3 Linear MMSE Estimation of Signals**

2.4 FIR Wiener Filter

2.5 Least Squares Optimal Filtering

- **2.5.1 LS Method**
- **2.5.2 LS Optimal Filtering**
- **2.5.3 Least Squares Orthogonality**

2.6 Introduction to Adaptive filtering

2.7 Adaptive Wiener Filter

2.8 The LMS Algorithm

2.9 Convergence Property of LMS Algorithm

2.10 Simplified LMS Algorithm

2.11 Applications

2.12 RLS Adaptive Filters

2.13 Adaptive Transversal Filters Using Least Squares Method

Random Process

References

. Sklar,B., Digital Communications , 2nd Ed. ,Prentice-Hall, 2001 .
. Couch, II, L.W., Digital and Analog Communication Systems, Seventh Ed. Pearson Prentice Hall, 2007.
.Therrien , C.W., Discrete Random Signals and Statistical Signal Processing. Prentice Hall,1992.
.Manolakis, D.G., Ingle , V.K., and Kogon ,S.M., Statistical and Adaptive Signal Processing , McGraw- Hill , 2000.
.Woods, J.W. and Stark, H., Probability and random Processes with Applications to Signal Processing ,Prentice Hall, 3rd Edition , 2002 .
.Leon-Garcia , A., Probability and Random Processes for Electrical Engineering , 3rd Edition, Prentice Hall .2008.
.Haykin, S.,, Adaptive Filter Theory, 4th ed. , Prentice Hall , 2002.

.

2. 1 Random Variable

- A **random variable** $X(A)$ represents the fundamental relationship between a random event A and a real number . For notational convenience, we usually designate the random variable by X .
- The random variable may be discrete or continuous.

2.1.1 Probability Distribution Function

- The distribution function $F_X(x)$ of the random variable X is given by
  $$F_X(x) = P(X \leqq x) \quad (1.1)$$
  where $P(X \leqq x)$ is the probability that the value taken by the random variable X is less than or equal to a real number x .
- $F_X(x)$ is also called the **cumulative distribution function (CDF)** .

- **The distribution function $F_X(x)$ has the following properties :**

  1. $0 \leqq F_X(x) \leqq 1$
  2. $F_X(x_1) \leqq F_X(x_2)$ if $x_1 \leqq x_2$
  3. $F_X(-\infty) = 0$
  4. $F_X(+\infty) = 1$

- **The probability distribution function (PDF) of the random variable X is defined as**

$$p_X(x) = d\,F_X(x)\ /\ dx \qquad (1.2)$$

The probability of the event $x_1 \leqq X \leqq x_2$ **equals**

$$P(x_1 \leqq X \leqq x_2) = P(X \leqq x_2) - P(X \leqq x_1)$$

$$= F_X(x_2) - F_X(x_1)$$

$$= \int p_X(x)dx \qquad (1.3)$$

- **The probability function has the following properties :**

  1. $p_X(x) \geqq 0$

  2. $\int p_X(x)dx = 1$

- **In the following , for ease of notation , we often omit the subscript X and write the PDF of a continuous random variable X simply as $p(x)$.**

  We will use the designation $p(X = x_i)$ for the probability of a discrete random variable X , where X can take on discrete values only .

- **Example 1.1 : Exponential Random Variable**

  $$P(X > x) = e^{-\lambda x} \qquad x \geqq 0$$

  The CDF of X is $F_X(x) = P(X \leqq x) = 1 - P(X > x)$

  $$= 0 \qquad x < 0$$

  $$1 - e^{-\lambda x} \qquad x \geqq 0$$

- **Example 1.1.2 : Uniform Random Variable**

$$p(x) = \begin{cases} 1/(b-a) & a \leqq x \leqq b \\ 0 & \text{otherwise} \end{cases} \quad (1.4)$$

Here we have

$E[X] = (b-a)/2$

$VAR[X] = (b-a)^2 / 12$

- **Example 1.1.3 Gaussian Random Variable**

$$p(x) = 1/\sqrt{(2\pi\sigma^2)}\ exp\{-(x-m)^2/2\sigma^2\} \quad (1.5)$$

2.1.2 Statistical Averages

- **The mean value m_X of a random variable X is defined by**

$$m_X = \mathrm{E}[X] = \int x \, p_X(x)\, dx \qquad (1.6)$$

- **The mean-square value of X is given by**

$$\mathrm{E}[X^2] = \int x^2 \, p_X(x)\, dx \qquad (1.7)$$

- **The variance of X is defined as**

$$\mathrm{var}(X) = \sigma_X^{\,2} = \mathrm{E}[(X - m_X)^2] = \int (x - m_X)^2 \, p_X(x)\, dx$$

$$(1.8)$$

- **The variance and the mean-square are related by**

$$\sigma_X^{\,2} = \mathrm{E}[X^2 - 2\, m_X X + m_X^{\,2}]$$

$$= \mathrm{E}[X^2] - m_X^{\,2} \qquad (1.9)$$

■ Chebyshev's Inequality

The variance $\sigma_X{}^2$ of a random variable *X* is a measure of the spread of the *x* values about their mean .

The Chebyshev inequality states that the *x*-values tend to cluster about their mean in the sense that the probability of a value not occurring in the near vicinity of the mean is small ; and it is the smaller the variance.

$$Pro\,[\;|\,x\text{-}m\,|\;\geqq\;\Delta\,]\;\leqq\;\sigma^2/\Delta^2 \qquad (1.10)$$

Appendix

1. We have

$$Pro\,[\,|x-m| \geqq \Delta\,] \leqq \sigma^2 / \Delta^2$$

For $\Delta = k\sigma$, then

$$Pro\,[\,|x-m| \geqq k\sigma\,] \leqq 1 / k^2$$

2. **Schwarz Inequality**

$$|(h, g)| = \|h\| \, \|g\|$$

3. Chi-squared density function

Foe a gaussian distribution function

$$f_X(x) = 1/\sqrt{(2\pi\sigma^2)}\ \ exp\{-x^2/2\sigma^2\}$$

define $Y = X^2$

The pdf of Y is given by

$$f_Y(y) = 1/\sqrt{(2\pi y\sigma^2)}\ \ exp\{-y^2/2\sigma^2\}.$$

This is a Chi-squared distribution function.

2.1.3 Conditional Probability and Bayes' Rule

- **Consider a combined experiment in which a joint event occurs with joint probability $P(A, B)$.**
- **Suppose that the event B has occurred and we wish to determine the probability of occurrence of the event A .**

This is called the conditional probability of the event A given the occurrence of the event B and is defined as

$$P(A \mid B) = P(A,B) / P(B) \qquad (1.11)$$

provided that $P(B) > 0$.

In a similar manner , the probability of the event B conditioned on the occurrence of the event A is defined as

$$P(B \mid A) = P(A,B) / P(A) \qquad (1.12)$$

provided that $P(A) > 0$.

These relations may also expressed as

$$P(A,B) = P(A \mid B)\, P(B) = P(\mathrm{B} \mid A)\, P(A) \qquad (1.13)$$

2.1.4 Conditional Probability Density

- **A pair of two different random variables ,**
  $X = (X_1, X_2)$, may be thought of as a vector-valued random variable. Its statistical description requires knowledge of the joint probability density $p(x_1, x_2)$ . .
- **A quantity that provides a measure for the degree of dependence of the two random variables on each other is the conditional probability density $p(x_1 \mid x_2)$ of x_1 given x_2 , and $p(x_2 \mid x_1)$ of x_2 given x_1 .**
- **Bayes' rule :**

$$p(x_1, x_2) = p(x_1 \mid x_2)\, p(x_2) = p(x_2 \mid x_1)\, p(x_1)$$

- **Two random variables are independent if they do not conditioned each other , that is , if**
  $p(x_2 \mid x_1) = p(x_2)$ and $p(x_1 \mid x_2) = p(x_1)$
  Then, $p(x_1, x_2) = p(x_1)\, p(x_2)$

2.1.5 Gaussian Random Variable

- **A Gaussian random variable *X* is one whose probability density function can be written in the general form**

$$p(x) = \{ 1 / \sqrt{(2\pi\sigma^2)} \} \; exp\, [- (x - m)^2 / 2\sigma^2)]$$

where *m* is the mean and σ^2 is the variance.

- **CDF of a Gaussian random variable**

$$F_X(x) = \int p(y)\, dy$$

$$= \int 1 / \sqrt{(2\pi)} \} \; exp\, [- t^2 / 2) \; dt$$

$$= 1 - Q[(x-m)/\sigma] \qquad (1.15)$$

where $Q(x) = 1 / \sqrt{(2\pi)} \} \int exp\, (- t^2 / 2) \; dt \qquad (1.16)$

Gaussian function with mean = 0 , variance = 1

$$g(x) = \{ 1/\sqrt{(2\pi)} \}\, exp\, (- x^2 / 2) \qquad \textbf{(1.17)}$$

Error function

$$erf\,(x) = \{ 2 / \sqrt{(\pi)} \} \int exp\, (- x^2)\; dt \qquad \textbf{(1.18)}$$

Complementary error function

$$erfc\,(x) = 1 - erf\,(x)$$

$$= 2 / \sqrt{(\pi)} \} \int exp\, (- x^2)\; dt \qquad \textbf{(1.19)}$$

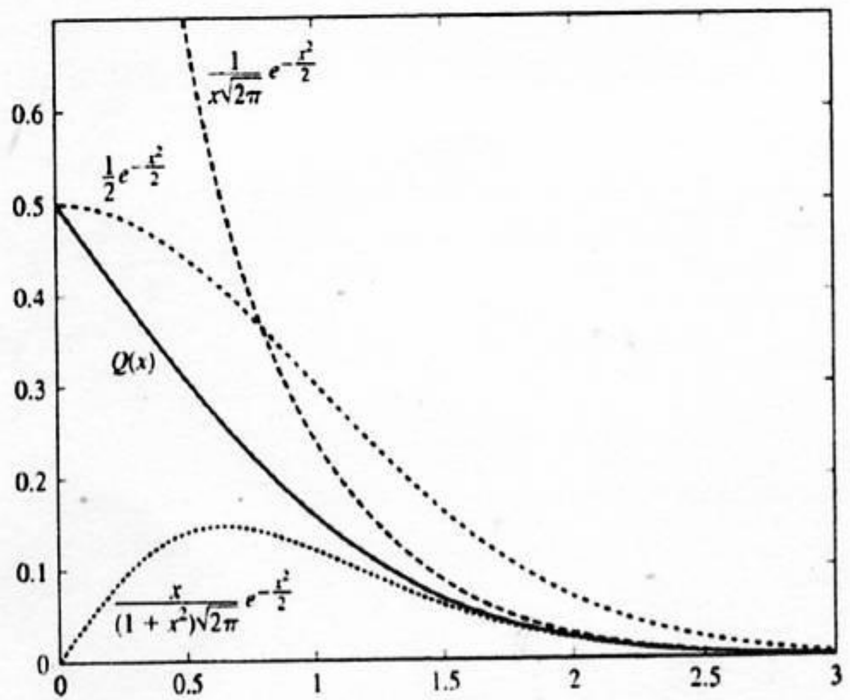
***Q*-function**

$$Q(x) = \{ 1/\sqrt{(2\pi)} \} \int exp\, (- x^2 / 2)\; dt \qquad \textbf{(1.20)}$$

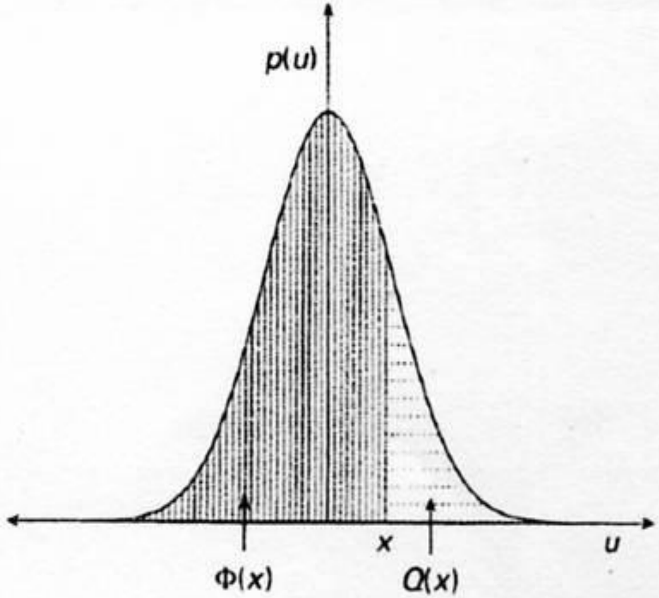
Relation between *erfc* (*x*) and *Q*- function :

$$Q\,(x) = ½\; erfc\,(x /\sqrt{2})$$

$$erfc\,(x) = 2\; Q(x \sqrt{2})$$



$\frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
$\frac{1}{2} e^{-\frac{x^2}{2}}$
$Q(x)$
$\frac{x}{(1+x^2)\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3



$p(u)$
x
u
$\Phi(x)$
$Q(x)$

2.1.6 Multiple Random Variables

- **Joint CDF**

The joint CDF of multiple random variable X_i , $i = 1, 2, \ldots, n$, is defined as

$$F(x_1, x_2, \ldots, x_n) = P(X_1 \leqq x_1, X_2 \leqq x_2, \ldots, X_n \leqq x_n)$$

$$= \int\int \ldots \int p(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n \qquad (1.21)$$

and $p(x_1, x_2, \ldots, x_n) = \qquad F(x_1, x_2, \ldots, x_n)$

(1.22)

- **The correlation between two random variables is defined as**

$$R_{XY} = E[XY] = \iint xy\, p_{xy}(x,y)\, dx\, dy = m_{XY} \qquad (1.23)$$

Two random variables , X and Y , are said to be uncorrelated if $m_{XY} = m_X\, m_Y$

- **The covariance between two random variables , X and Y , is defined as**

$$\sigma_{xy} = \mathrm{Cov}(X,Y) = E[(X - m_x)(Y - m_y)]$$

$$= \iint (x - m_x)(y - m_y)\, p_{XY}(x,y)\, dxdy \qquad (1.24)$$

The correlation coefficient is defined as

$$\rho_{XY} = \sigma_{xy} / \sigma_x \sigma_y \quad , \quad -1 < \rho_{XY} < 1 \qquad (1.25)$$

- **If X and Y are statistically independent, then $\rho_{XY} = 0$**

  Note that random variables, X and Y, are independent if

  $$P(X = x_i,\ Y = y_i) = P(X = x_i)\,P(Y = y_i)$$

- # Gaussian Bivariate Distribution

The bivariate Gaussian PDF of two random variables, X and Y, is expressed as

$$p_{XY}(x,y) = \{ 1 / [(2\pi\, \sigma_x\sigma_y)\sqrt{(1-\rho^2)}] \}\ exp \{ -(1/2)/(1-\rho^2) [(x-m_x)^2/\sigma_x^2 + (y-m_y)^2/\sigma_y^2 - 2\rho(x-m_x)(y-m_y)/\sigma_x\sigma_y] \} \qquad (1.26)$$

If they are independent, then

$$p_{XY}(x,y) = \{ 1 / (2\pi\, \sigma_x\sigma_y) \}\ exp \{ -(1/2) [(x-m_x)^2/\sigma_x^2 + (y-m_y)^2/\sigma_y^2] \} \qquad (1.27)$$

2.1.7 Sum of iid Random Variables and the Central Limit Theorem

- **Suppose that X_i, $i = 1, 2, \ldots n$, are statistically independent and identically distributed (iid) random variables, each having a finite mean m_x and a finite variance σ^2.**

- **Let Y_n be defined as the normalized sum, called the sample mean :**

$$Y_n = (1/n)\Sigma X_i \qquad (1.30)$$

The mean of Y_n is

$$E[Y_n] = m_y = (1/n)\Sigma E[X_i]$$
$$= m_x \qquad (1.31)$$

The variance of Y_n is $\sigma_y^{\,2} = E\,(Y_n^{\,2}) - m_y^{\,2}$

$$= (\,1/\,n^2\,)\,\Sigma\,\Sigma\,\mathrm{E}(\,X_i\,X_j\,)$$

$$= (1/n\,)\sigma_x^{\,2} \qquad (1.32)$$

Central Limit Theorem

Define the normalized random variable

$$Z_n = (Y_n - m_n) / \sigma_y = \Sigma (X_i - m)/ (\sigma_x \sqrt{n}) \quad (1.33)$$

Then the random variable Z_n has a distribution that is asymptotically unit normal.

That is, as n becomes large , the distribution of Z_n approaches that of a zero-mean Gaussian random variable with unit variance.

2.1.8 Transformation of Random Variables

- **Let X_1 and X_2 be continuous random variables with a joint PDF $f_{X1, X2}(x_1, x_2)$, and consider the transformation defined by**

$$y_1 = h_1(x_1, x_2) \quad \text{and} \quad y_2 = h_2(x_1, x_2)$$

Which are assumed to be one-to-one and continuously differentiable.

The Jacobian of this transformation is defined by the matrix determinant

$$J(\qquad) = \det \qquad \neq 0 \qquad (1.34)$$

The joint pdf of *Y*1 and *Y*2 is given by

$$f_{Y1\ Y2}\ (y_1, y_2) = f_{X1\ X2}(x_1, x_2)\ \left| J(\qquad) \right| \qquad (1.35)$$

- **Example 1.1: Rayleigh distribution**

Let $Y = \sqrt{(X_1^2 + X_2^2)}$
where X_1 **and** X_2 **are independent zero- mean Gaussian random variables.**

$f_{X,\,X}(x_1, x_2) = (1/2\pi\sigma^2)\ exp\,[-\ (x_1^2 + x_2^2) / 2\,\sigma^2\,]$

we will transform the two Gaussian random variables from Cartesian to polar coordinates.

Denote that $R = \sqrt{(X_1^2 + X_2^2)}$

and $\Theta = tan^{-1}(X_2 / X_1)$

The inverse transformation is

$X_1 = R\ cos\Theta\ ,\ X_2 = R\ sin\Theta\ ,$

The Jacobian is then calculate as

$$J = \det \qquad = \det$$

$$= r$$

The joint pdf of R and Θ is then

$$f_{R\,\Theta}(r, \theta) = f_{X1,\,X2}(x_1, x_2)\ \ J$$

$$= (r/2\pi\sigma^2)\ exp\ [-(x_1^2 + x_2^2)/2\sigma^2]$$

$$= (r/2\pi\sigma^2)\ exp\ [-\ r^2/2\sigma^2]$$

$$r \geqq 0\ ,\ \ 0 \leqq \theta \leqq 2\pi \qquad (1.36)$$

The marginal distribution of R and Θ are then given by

$$f_R(r) = (r/\sigma^2)\exp(-r^2/2\sigma^2) \qquad r \geqq 0 \qquad (1.37)$$

and $$f_\Theta(\theta) = 1/(2\pi)\,, \quad 0 \leqq \theta \leqq 2\pi \qquad (1.38)$$

Note that $f_R(r)$ is the PDF of a Rayleigh random variable .
It is also denoted as Rayleigh distribution.

- **Example 1.2 : Rician distribution**

If X_1 and X_2 are two independent Gaussian random variables with means m_1 and m_2 , respectively, and a common variance σ^2 , then the new random variable

$$R = \sqrt{(X_1^2 + X_2^2)}$$

has a Rician distribution.

The pdf of R is given by

$$f_R(r) = (r/\sigma^2)\, exp\{-(r^2 + s^2)/2\sigma^2)\, I_0(r s/\sigma^2)\} \qquad r \geqq 0 \qquad (1.39)$$

where $I_0(.)$ is the modified Bessel function of zero order , and $s = \sqrt{(m_1^2 + m_2^2)}$, $K = s^2/2\sigma^2$ is denoted as Rician factor

•

Appendix: Complex Random Variables

- $R_{XX} = E[\,XX^H]$

  where H denotes the Hermitian transpose operation.

- $R_{XY} = E[\,XY^H]$

- $C_{XY} = E[(X - m_X)(Y - m_Y)^H]$

2.2 Random Signal and Random Process

2.2.1 Random Vector

- **For a vector of random variables $X = [\, X_1 \; X_2 \ldots \; X_N \,]^T$,**

  we can construct a corresponding mean vector that is a column vector of the same dimension and whose components are the means of the elements of X .

  That is, $m_x = E[X] = \{ E[X_1] \; E[X_2] \ldots \; E[X_N] \}^T$ (1.44)

  The correlation matrix is defined as

  $$R_{XX} = E[\, X X^T \,] \tag{1.45}$$

  Similarly , the covariance matrix is defined as

  $$C_{XX} = E\,[(\, x - m_x\,)\,(x - m_x\,)^T\,] \tag{1.46}$$

- **Theorem :**

  Correlation matrices and covariance matrices are symmetric and positive definite.

- **Correlation between two random vectors X and Y is given by**

  $R_{XY} = E[\, XY^T \,]$ **(1.47)**

- **Joint Expectations for Two Random vectors**

(a) For two random vectors, X and Y,

the cross-correlation matrix is defined as

$$R_{XY} = E[XY^T] \qquad (1.48)$$

and the cross-covariance matrix is define as

$$C_{XY} = E[(X - m_x)(Y - m_y)^T] \qquad (1.49)$$

where m_x and m_y are mean vectors of X and Y, respectively.

It can be shown that $R_{XY} = C_{XY} + m_X m_Y^T$ (1.50)

(b) The random vectors X and Y are said to be uncorrelated if $R_{XY} = m_X m_Y^T$; or , equivalently, the cross-covariance $C_{XY} = 0$

(c) Two vectors are said to be orthogonal if $R_{XY} = 0$, or the cross-correlation is zero

2.2.2 Random Process and Statistical Average

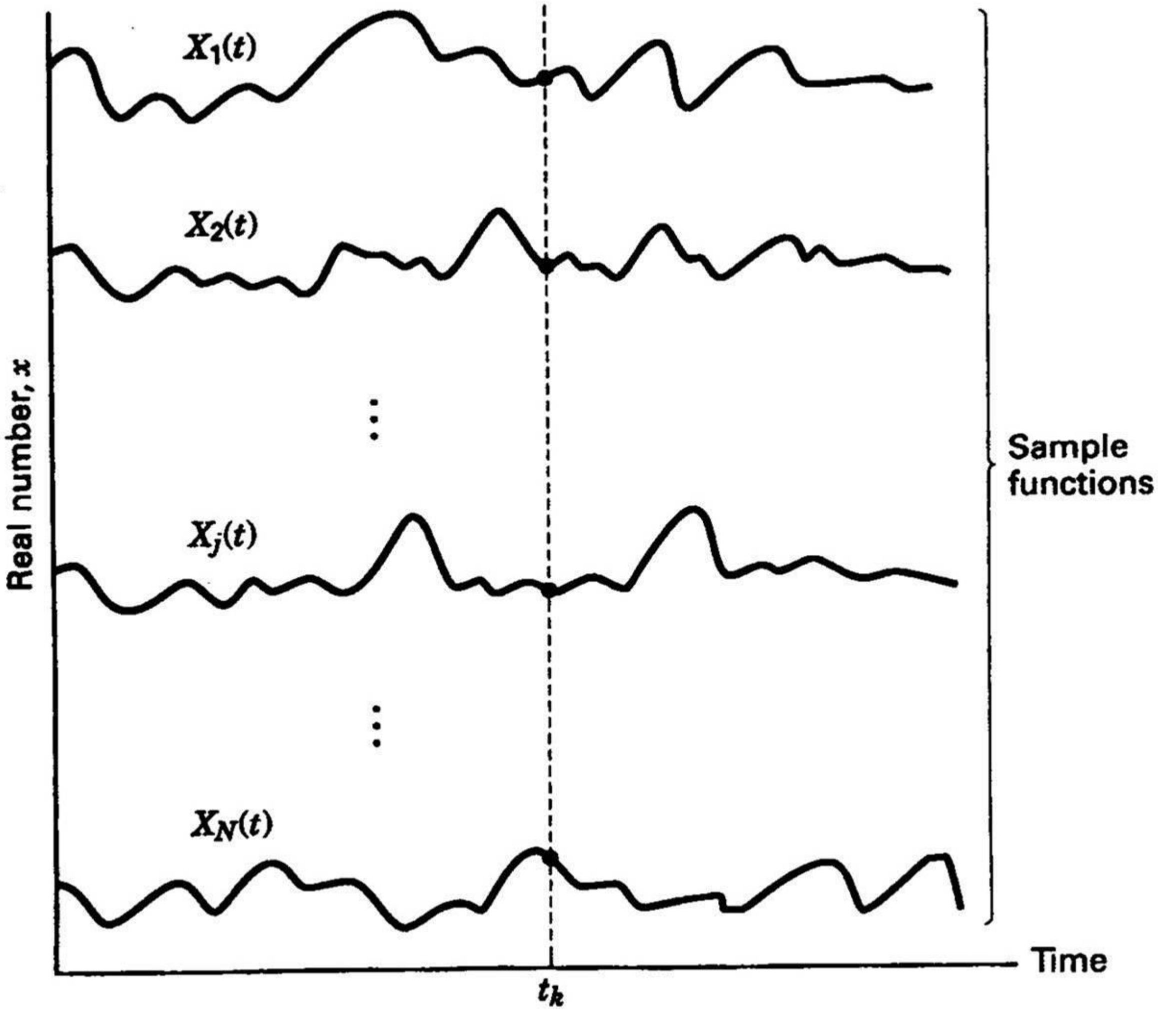
2.2.2.1 Random Process

- **A random process (or stochastic process) $X(A,t)$ can be viewed as a function of two variables : an event A and time t .**
- **Fig.1 illustrates a random process. In the figure, there are N sample functions of time, $\{ X_j(t) \}$.**

  Each of the sample functions can be regarded as the output of a different noise generator. For a specific event A_j, there is a a single time function $X(A_j ,t) = X_j(t)$, i.e. a sample function.

  The totality of all sample functions is called an ensemble.

  For a specific time t_k, $X(A_j ,t) = X_j(t_k)$ is simply a number.
- **For notational simplicity , we shall designate the random process by $X(t)$.**



Random process.

2.2.2.2 Statistical Averages

- **A random process whose distribution functions are continuous can be described statistically with a probability density function (PDF). In general , the form of the PDF of a random process will be different for different times.**
- **The mean of a random process**

$$E\{X(t_k)\} = \int x\, p_X(x)dx \equiv m_X \qquad (1.78)$$

where $X(t_k)$ is the random variable obtained by observing the random process at time t_k . The pdf of $X(t_k)$ is designated as $p_X(x)$.

- **The autocorrelation function of a random process is defined as**

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] \qquad (1.79)$$

The autocorrelation function is a measure of the degree to which two time samples of the same random process are related.

- **In general, the *n*th moment is defined as**

$$E\{X^n\} = \int x^n\, p_X(x)\, dx \qquad (1.80)$$

- **E{ $(X - m_X)^n$ } is called the *n*th central moment , and when n=2 , the central moment is called the variance of the random process , denoted by σ_X^2.**

$$\sigma_X^2 = \int (x - m_X)^2\, p_X(x)\, dx \qquad (1.81)$$

- **Stationary Random Process (in the strict sense) :**
The statistics of a stationary random process are invariant to any translation of the time axis. That is,

$$p(x(t_1), x(t_2), \ldots, x(t_n)) = p(x(t_1+\tau), x(t_2+\tau), \ldots, x(t_n+\tau)) \qquad (1.82)$$

- **Wide-Sense Stationary (WSS) Random Process**

A random process is said to be wide-sense stationary (WSS) if two of its statistics , mean and autocorrelation, are invariant to a time shift. That is,

$$E\{ X(t) \} = m_X = \text{a constant}$$

and $$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau) \quad (1.83)$$

where $\tau = (t_1 - t_2)$ **(1.84)**

- **Properties of Autocorrelation of a Real-Valued WSS Random Process**

1. $R_X(\tau) = R_X(-\tau)$
2. $|R_X(\tau)| \leqq R_X(0)$ for all τ
3. $R_X(\tau) \longleftarrow\longrightarrow S_X(f)$ = power spectral density
4. $R_X(0)$ = $E[X^2(t)]$ = average power of the signal

- **Stationary and Ergodicity**

  A stationary random process is said to be ergodic if time averages of a sample function are equal the corresponding ensemble average (or expectation) at a particular point in time.

That is,

$$m_X = Lim\ (\ 1/T\ ) \int X(t)\ dt \qquad (1.85)$$

and $$R_X = Lim\ (1/T\ ) \int X(t\ )\ X(\ t+\tau)\ dt \qquad (1.86)$$

Example : Gaussian Random Process

- A random process $x(t)$ is said to be Gaussian if the random variables

  $x_1 = x(t_1)$, $x_2 = x(t_2)$, ... , $x_n = x(t_N)$

  have an N-dimensional Gaussian PDF for any N and x_1 , x_2 ,..., x_N
- The N-dimensional Gaussian PDF is

$$f_X(x) = \{1/(2\pi)^{N/2} \mid \text{Det C} \mid^{1/2}\}$$
$$exp\{-(1/2)[(x-m)^T C^{-1}(x-m)]\}$$

  where m is the mean vector , C is the covariance matrix of x .
- For a wide-sense stationary process , $m_i = E[x(t_i)] = m_j = E[x(t_j)]$, and the element of the covariance matrix become

$$c_{ij} = E[(x_i - m_i)(x_j - m_j)] = E[(x_i - m_i)]\ E[(x_i - m_j)]$$

- If , in addition, the x_i happen to be uncorrelated (e.g., white noise) ,

$$E[(x_i - m_i)] = E[(x_i - m_j)] \quad \text{for} \quad i \neq j$$

  then $c_{ij} = \sigma^2$, $c_{ij} = 0$ for $i \neq j$

2.2.2.3 Power Spectral Density

- **Definition of power spectral density (PSD) :**

For a random process $X(t)$, define a truncated version of the random process as

$$X_a(t) = \begin{cases} X(t) & |t| \leqq a \\ 0 & |t| > a \end{cases} \qquad (1.87)$$

$\geqq$

The energy of this random process is

$$E_{Xa} = \int X^2(t)\,dt = \int X_a{}^2(t)\,dt \qquad (1.88)$$

Hence, the time-average power is

$$P_{Xa} = (1/2a)\int X_a{}^2(t)\,dt = (1/2a)\int X_a{}^2(f)\,df \qquad (1.89)$$

- **The last quantity is obtained using Parseval's theorem. The quantity $X_a(f)$ is the Fourier transform of $X_a(t)$.**
- **Note that P_{Xa} is a random variable and so to get the ensemble average power , we must take an expectation ,**

$$P_{Xa} = E[P_{Xa}] = (1/2a) \int E[\,|\,X_a(f)\,|^2\, df \qquad (1.90)$$

The power in the (untruncated) random process $X(t)$ is then found by passing to the limit $a \rightarrow \infty$,

$$P_{Xa} = \text{Lim}\,(1/2a) \int E[\,|\,X_a(f)\,|^2\,]\, df$$

$$= \int \text{Lim}\,(1/2a)\; E[\,|\,X_a(f)\,|^2\,]\, df \qquad (1.91)$$

Define $S_X(f) = \text{Lim}\ (1/2a)\ E[\ |\ X_a(f)\ |^2\ ]$ **(1.92)**

Then, the average power in the process can be expressed as

$$P_X = \int S_X(f)\, df \qquad (1.93)$$

$S_X(f)$ **is denoted as power spectral density of the random process** $X(t)$ **.**

- **Note : Parseval's energy theorem**

The energy of a non-periodic signal $g(t)$ **is equal to the total area under the curve of the energy density spectrum** $S_g(f)$ **, where**

$$E_g = \int |\ g(t)\ |^2\ dt = \int |\ G(f)\ |^2\ df \qquad (1.94)$$

and $g(t)$ $G(f)$ **(1.95)**

- **Wiener-Khinchine Relation :**

  For a wide- sense stationary random process $X(t)$ whose autocorrelation function is given by $R_X(\tau)$, the power spectral (PSD) of the process is

$$S_X(f) = F\{R_X(\tau)\} = \int R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{1.96}$$

  In other words, the autocorrelation function and power spectral density for a Fourier transform pair.

2.4.4 Cross Correlation

- **Definition : The cross correlation between two random processes $X(t)$ and $Y(t)$ is defined as**

$$R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)] \tag{1.97}$$

- **Two random processes $X(t)$ and $Y(t)$ are jointly stationary if both $X(t)$ and $Y(t)$ are individually stationary , and the cross correlation $R_{XY}(t_1, t_2)$ depends only on $\tau = (t_1 - t_2)$.**

It follows that

$$R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2) = R_{XY}(\tau)$$

- **Example :**

If two random processes $X(t)$ and $Y(t)$ are jointly stationary and $Z(t) = X(t) + Y(t)$ then the autocorrelation of Z(t) is

$$R_Z(t+\tau, t) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{XY}(-\tau)$$

2.2.3 Response of a Linear Time-Invariant System to Random Signals

- **Consider a linear time-invariant (LTI) system characterized by its impulse response $h(t)$, *or*, equivalently, by its frequency response $H(f)$, where $h(t)$ and $H(f)$ are a Fourier transform pair. That is,**

$$H(f) = \int h(t)\, e^{-j2\pi f t}\, dt \qquad (1.101)$$

$$h(t) = \int H(f)\, e^{j2\pi f t}\, df \qquad (1.102)$$

- **Let $x(t)$ be the input signal to the system and let $y(t)$ denote the output signal. Then $y(t)$ can be expressed in terms the convolution integral**

$$y(t) = \int h(\tau)\, x(t - \tau)\, d\tau \qquad (1.103)$$

- **Now , suppose that x (t) is a sample function of a stationary stochastic process X (t). Then, the output y (t) is a sample function of a stochastic process Y (t) . The statistical averages are given as follows.**

The mean value of Y (t) is

$$m_Y(\mathrm{t}) = E[Y(\mathrm{t})] = \int h(\tau)\, E[X(\mathrm{t}-\tau)]\, d\tau$$

$$= m_x \int h(\tau)\, d\tau = m_x H(0) \qquad (1.104)$$

where $H(0)$ is the frequency response of the linear system at $f = 0$.

The autocorrelation function of the output is

$$\Psi_{yy}(t_1, t_2) = (1/2)\ E[\, Y_{t1}\, , Y_{t2}^{*}\,]$$

$$= (1/2) \int\int h(\beta)\, h^*(\alpha)\, E[X(t-\beta)X^*(t-\alpha)]\, d\alpha\, d\beta$$

$$(1.105)$$

After some mathematical manipulations, we finally obtain

$$\psi_{yy}(\tau) = \iint h(\beta)\, h^*(\alpha)\, \psi_{xx}(\tau+\alpha-\beta)\, d\alpha\, d\beta$$

- By evaluating the Fourier transform of both sides of the above equation , we obtain the power spectral density of the output process in the form

$$\Phi_{yy}(f) = \Phi_{xx}(f)\, \left| H(f) \right|^2$$

When the autocorrelation function $\psi_{yy}(\tau)$ is desired, it can be evaluated by

$$\psi_{yy}(\tau) = \int \Phi_{yy}(f)\, e^{j2\pi f\tau}\, df$$

and $\psi_{yy}(0) = \int \Phi_{xx}(f)\, \left| H(f) \right|^2\, df$

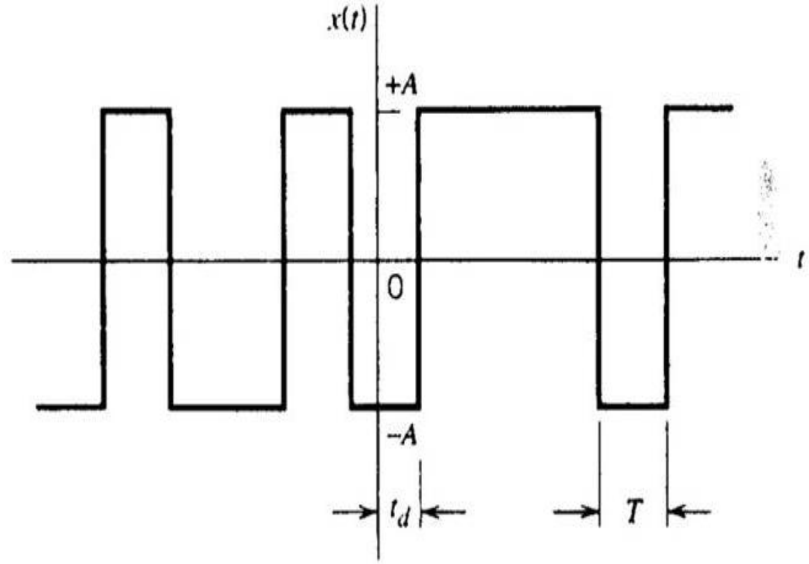
Example : Random Binary signal

The figure shows the sample function $x(t)$ of a random process $X(t)$ consisting of a random binary sequence of binary symbols , 1 and 0 . The following assumptions are made :

1. The symbols 1 and 0 are represented by rectangular pulses of amplitudes +A and –A , respectively.
2. The pulses are not synchronized , so the staring time of the first complete pulse for positive time is equaly to lie between 0 and T. Thus τ_d is a random variable uniformly distributed between *0* and *T* .
3. The amplitude level –A and +A occur with equal probability.

   Thus E[X(t)] = 0 for all *t* .

Consider the first case when $|t_k - t_j| > T$, *the random variables* $X(t_k)$ and $X(t_j)$ occur in different pulse intervals and are ,therefore , independent . Thus we have $E[X(t_k)\,X(t_j)] = E[X(t_k)]\,E[X(t_j)] = 0$



Sample function of random binary wave.

Consider next the case when $| t_k - t_i | < T$, with $t_k = 0$, $t_i < t_k$, *or* $t_i > t_k$.
In such a situation, we can see that ,from the figure , that the random variables $X(t_k)$ and $X(t_i)$ occur in the same pulse interval If and ony if the delay τ_d satisfies the condition

$$\tau_d < T - | t_k - t_i |$$

Thus we obtain the conditional expectation

$$E[X(t_k)\,X(t_i) \mid \tau_d] = \begin{cases} A^2 & \tau_d < T - | t_k - t_i | \\ 0 & \text{elsewhere} \end{cases}$$

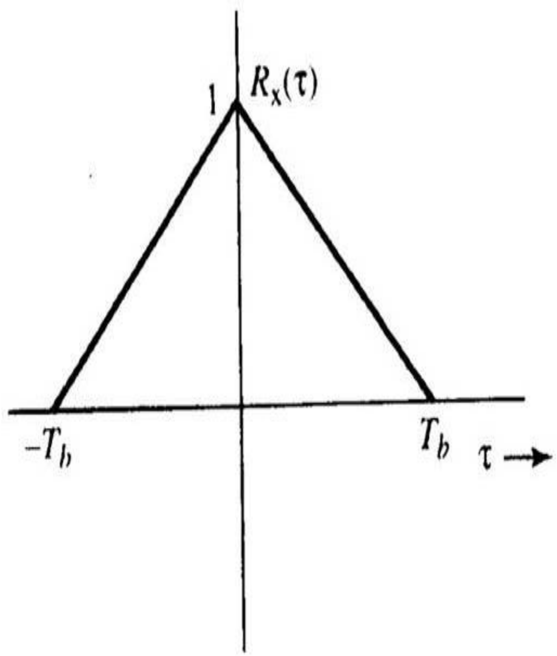
Averaging this result over all possible values of τ_d , we get

$$E[X(t_k)\,X(t_i)] = \int (A^2 / T)\, d\tau_d$$

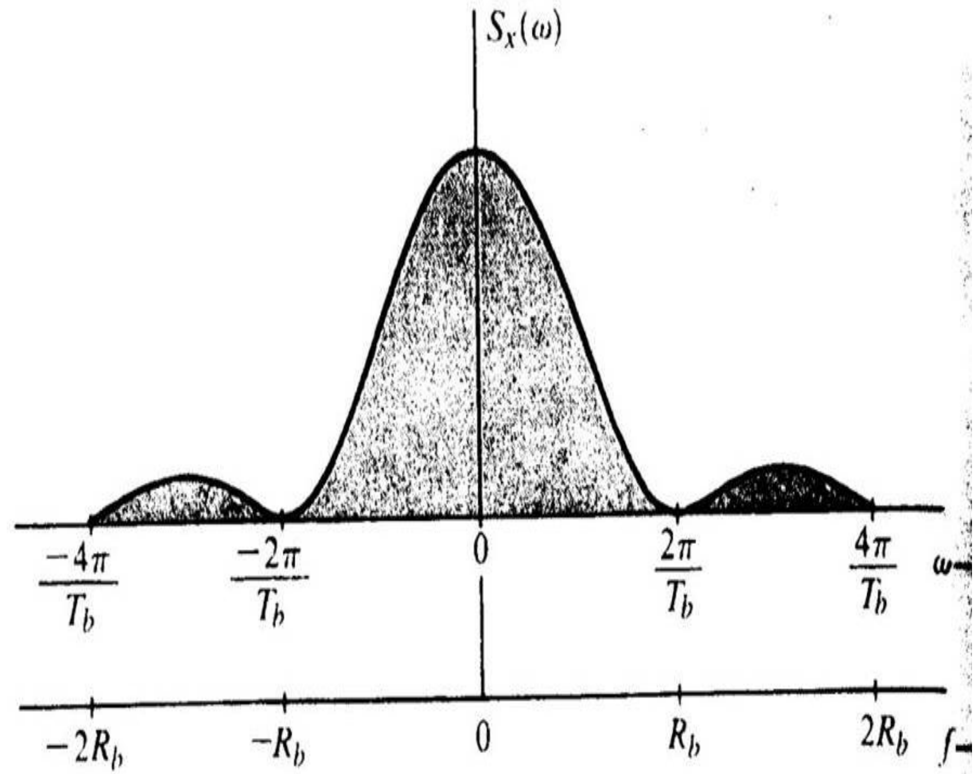
$$= A^2 \,(1 - | t_k - t_i | / T) \quad , \qquad | t_k - t_i | < T$$

By same reasoning for any other values of t_k , we conclude that the autocorrelation function of a random binary wave can be expressed as

$$R_X(\tau) = \begin{cases} A^2\,(1 - | \tau | / T) & | \tau | < T \\ 0 & | \tau | \geqq T \end{cases}$$

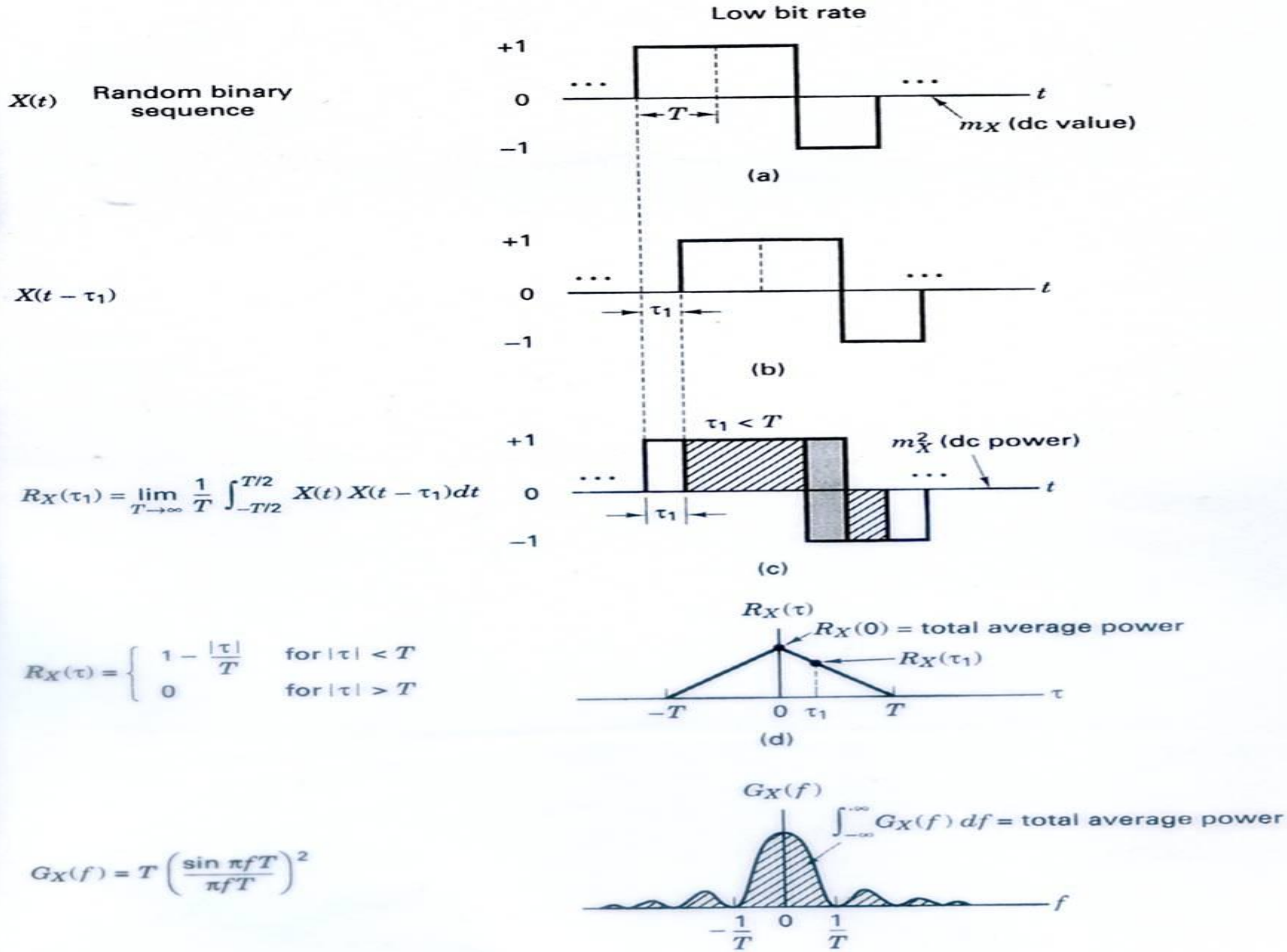

1
$R_x(\tau)$
$-T_b$
T_b
$\tau \rightarrow$

(c)


$S_x(\omega)$
$\frac{-4\pi}{T_b}$
$\frac{-2\pi}{T_b}$
0
$\frac{2\pi}{T_b}$
$\frac{4\pi}{T_b}$
$\omega \rightarrow$
$-2R_b$
$-R_b$
0
R_b
$2R_b$
$f \rightarrow$

(d)

- **Random Polar Binary Signal**



Autocorrelation and power spectral density.

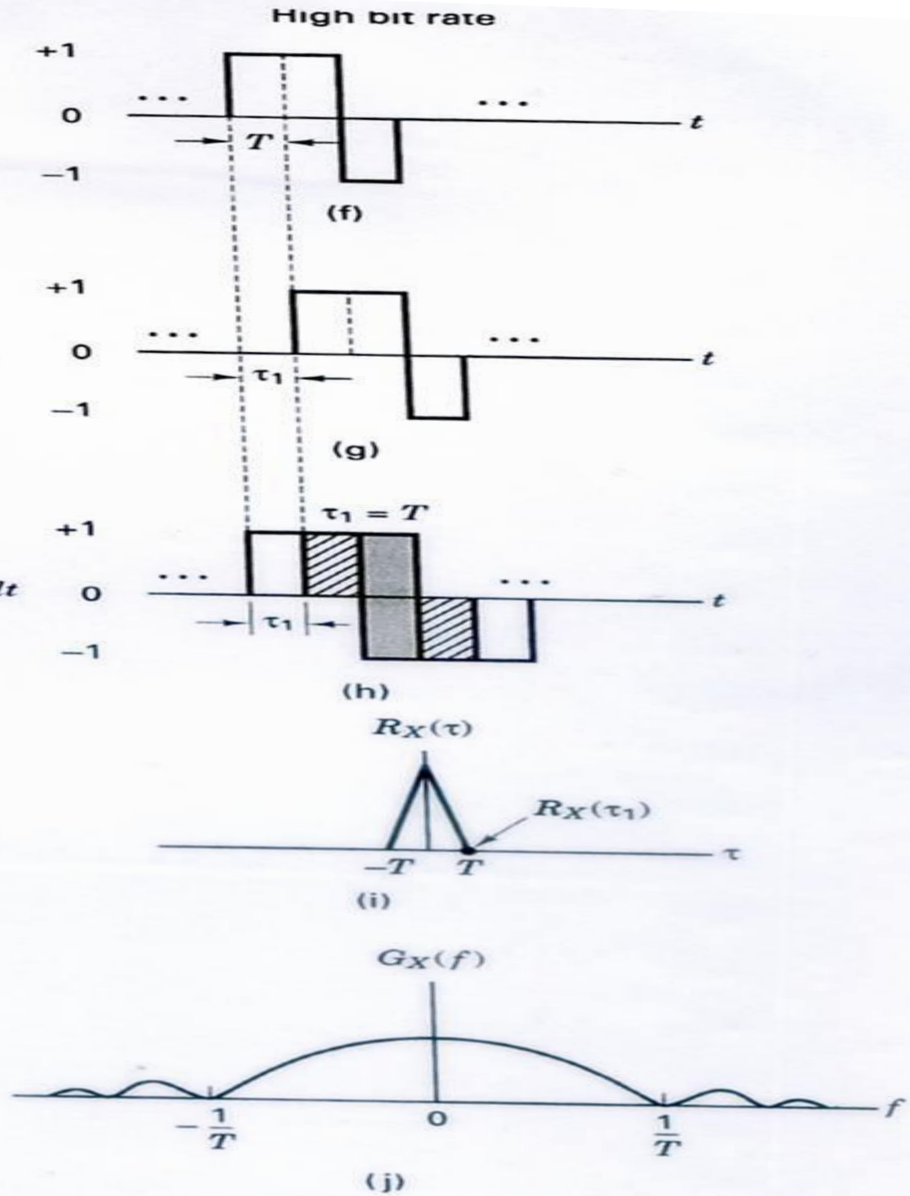
$X(t)$ Random binary sequence

$X(t-\tau_1)$

$$R_X(\tau_1) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,X(t-\tau_1)\,dt$$

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & \text{for } |\tau| < T \\ 0 & \text{for } |\tau| > T \end{cases}$$

$$G_X(f) = T\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$



continued

2.2.4 Bandpass Random Process

- **We define a bandpass (or narrowband) process as a real , zero-mean , and WSS random process by**

$$X(t) = \mathrm{Re}\,[\, g(t)\, exp\, (\, j2\pi f_0 t + \theta_c)\,]$$
$$= X_i(t) \cos\, (2\pi f_0 t + \theta_c) - X_q(t) \sin\, (2\pi f_0 t + \theta_c)$$

where $X_i(t)$ and $X_q(t)$ are denoted as the equivalent lowpass in-phase component and quadrature component , respectively, and θ_c is an independent random variable uniformly distributed over (0 , 2π) .

The lowpass equivalent process is given by

$$g(t) = X_i(t) + j\, X_q(t)$$

The constant θ_c is often called the random start-up phase.

- **We can show that [Couch, pp. 446-452]**

1. $g(t)$ is a complex WSS baseband process .

2. $X_i(t)$ and $X_q(t)$ are jointly WSS zero-mean random processes .

3. $X_i(t)$ and $X_q(t)$ have the same power spectral density.

$$S_{Xi}(f) = S_{Xq}(f) = \begin{cases} [S_X(f - f_c) + S_X(f + f_c)] & |f| < B \\ 0 & \text{otherwise} \end{cases}$$

where B is the bandwidth of g(t).

4. Autocorrelation function

$$R_X(\tau) = \tfrac{1}{2}\ \mathrm{Re}\,\{R_g(\tau)\, exp\,(j\, 2\pi f_0 \tau)\}$$

5. Power spectral density

$$S_X(f) = \tfrac{1}{4}\ [S_g(f - f_c) + S_g(-f - f_c)]$$

Example : Filtered White Gaussian Noise

White Gaussian noise with power spectral density of $N_0 / 2$ passes through an ideal bandpass filter with transfer function

$$H(f) = \begin{cases} 1 & |f - f_0| < B \\ 0 & \text{otherwise} \end{cases}$$

where $B < f_0$.

The output , called filtered Gaussian white noise , is denoted by $X(t)$.

The power spectral density of the filtered noise will be

$$S_X(f) = (N_0 / 2)\, |H(f)|^2$$

The filtered white Gaussian noise can also expressed as

$$X(t) = X_i(t) \cos(2\pi f_0 t) - X_q(t) \sin(2\pi f_0 t)$$

where $X_i(t)$ and $X_q(t)$ are the in-phase and quadrature components of $X(t)$, respectively , and are lowpass processes .

The power spectral density of the lowpass- equivalent processes are given by

$$S_{Xi}(f) = S_{Xq}(f) = \begin{cases} N_0 & |f| < B \\ 0 & \text{otherwise} \end{cases}$$

and $$S_g(f) = \begin{cases} 2N_0 & |f| < B \\ 0 & \text{otherwise} \end{cases}$$

Power of the bandpass Gaussian noise = $2 N_0 B$

Example : Power Spectral Density of BPSK signal

The BPSK signal can be expressed by

$$v\,(t) = x(t)\cos(2\pi f_0 t + \theta_c)$$

where x(t) represents the polarity binary data and θ_c is the random start-up phase.

The PSD of v (t) is found by

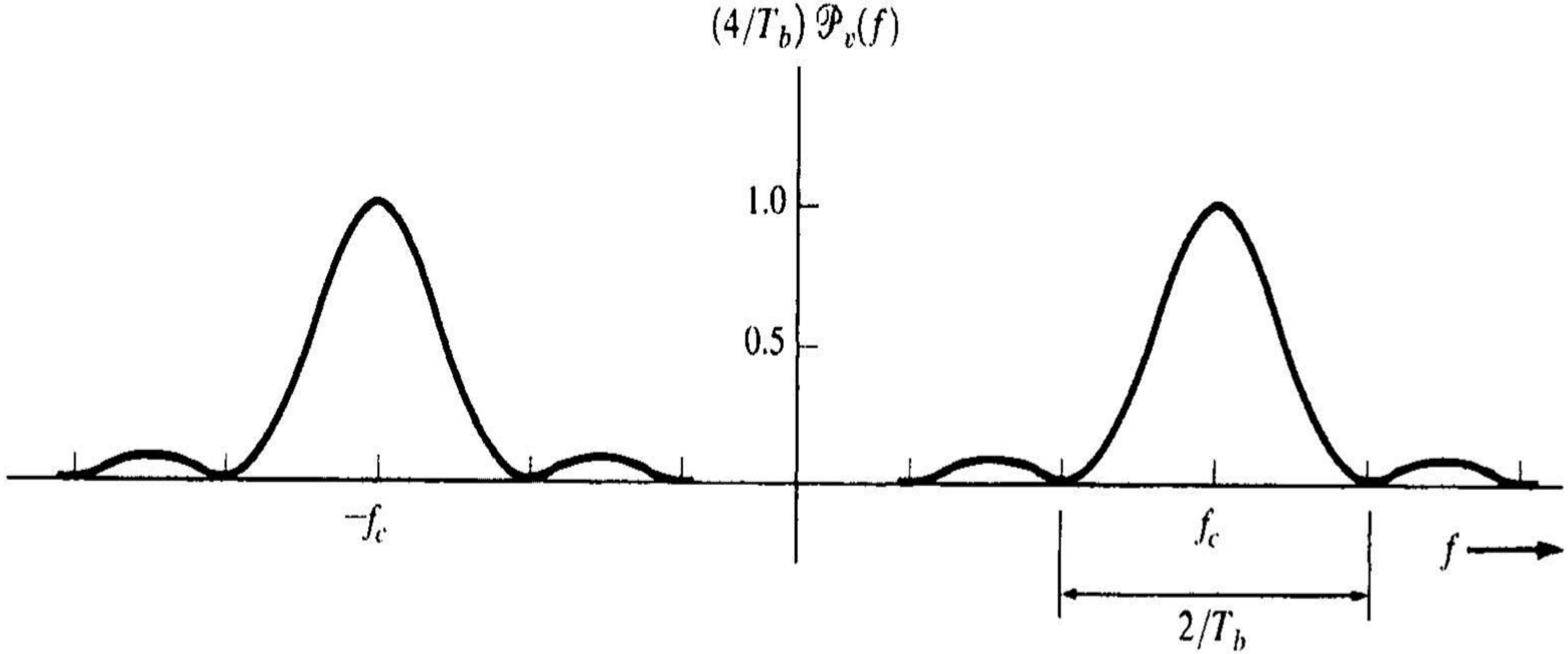
$$S_v(f) = \tfrac{1}{4}\,[S_x(f - f_c) + S_x(-f - f_c)]$$

The PSD of the polar baseband signal with equally likely binary data is given by

$$S_x(f) = T_b\,(\sin \pi f T_b \;/\; \pi f T_b)^2$$

We then obtain the PSD for the BPSK signal

$$S_v(f) = (1/4)\,T_b\,\{\,[\sin \pi(f - f_c)T_b \;/\; \pi(f - f_c)T_b]^2 + [\sin \pi(f + f_c)T_b \;/\; \pi(f + f_c)T_b]^2\,\}$$



Power spectrum for a BPSK signal.

Exercise #2

The PDF of a Rayleigh –distributed random variable X is given by

$$p(x) = \left(x/\sigma^2 \right) \exp\left(-x^2 / 2\sigma^2 \right) \qquad x \geqq 0$$

Find the mean and variance of X.

Answer : **mean =** $\sigma / \sqrt{(\pi/2)}$

variance = $\sqrt{(2-\pi/2)}\,\sigma$

2.2.5 Markov Processes

- **Markov Process :**

A random process, X(t), is said to be a Markov process if for any time instants, $t_1 < t_2 < \ldots < t_n < t_{n+1}$, the random process satisfies

$$F_X(X(t_{n+1}) \leqq x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1)$$

$$= F_X(X(t_{n+1}) \leqq x_{n+1} \mid X(t_n) = x_n) \qquad (1.98)$$

The Markovian property states that given the present, the future is independent of the past .

In other words, the future of the random process depends only on where it is now and not on how it got there.

Example #1 Sinusoidal wave with random phase

$X(t) = A \cos(2\pi f_c t + \Theta)$

where A is a constant and Θ is a random variable with uniform pdf

over the interval $[-\pi, \pi]$, i.e.,

$$f_{\Theta}(\theta) = \begin{cases} 1/2\pi & , \ -\pi \leqq \theta \leqq \pi \\ 0 & , \ \text{elsewhere} \end{cases}$$

1. Find the autocorrelation function of X(t)

   Ans. $R_X(\tau) = (A^2/2) \cos(2\pi f_c)$

2. Find the power spectral density of X(t)

   Ans. $S_X(f) = (A^2/2) [\delta(f - f_c) + \delta(f - f_c)]$

Example # 2

If $Y(t) = X(t) \cos(2\pi f_c t + \Theta)$

where $X(t)$ is a stationary random process and Θ is a randomvariable with uniform pdf over the interval $[-\pi, \pi]$

Find the autocorrelation function and power spectral density of $X(t)$.

Ans. $R_Y(\tau) = (1/2) R_X(\tau) \cos(2\pi f_c \tau)$

$S_Y(f) = (1/2) [S_X(f - f_c) + S_X(f + f_c)]$

Example #3

A stationary Gaussian process $X(t)$ with zero-mean and PSD $S_X(f)$ is applied to a linear filter whose impulse response is a rectangular function of time , duration = T , height = 1/ T .

Y(t) is the output at time t .

1. find the mean and variance of Y(t)
2. what is the probability density function of Y.?
3. Find the output power spectral density.

Ans. $H(f) = \exp(-j \pi f T) \sin(\pi f T)/(\pi f T)$

$S_{YY}(f) = [\sin^2(\pi f T) / (\pi f T)^2] S_{XX}(f)$