Information Theory (H.-M. Hang; 2009/9/28)

(Cover and Thomas, Chaps 2 and 4)

Homework #1

Due Date: October 6, 2009

(1)

2. **Entropy of functions.** Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if

   (a) $Y = 2^X$?

   (b) $Y = \cos X$?

(2)

12. **Example of joint entropy.** Let $p(x, y)$ be given by

X \ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

   Find

   (a) $H(X), H(Y)$.

   (b) $H(X \mid Y), H(Y \mid X)$.

   (c) $H(X, Y)$.

   (d) $H(Y) - H(Y \mid X)$.

   (e) $I(X; Y)$.

   (f) Draw a Venn diagram for the quantities in (a) through (e).

(3)

14. **Entropy of a sum.** Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.

   (a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of *independent* random variables adds uncertainty.

   (b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

   (c) Under what conditions does $H(Z) = H(X) + H(Y)$?

(4)

38. **The value of a question** Let $X \sim p(x)$, $x = 1, 2, \ldots, m$. We are given a set $S \subseteq \{1, 2, \ldots, m\}$. We ask whether $X \in S$ and receive the answer

$$Y = \begin{cases} 1, & \text{if } X \in S \\ 0, & \text{if } X \notin S. \end{cases}$$

Suppose $\Pr\{X \in S\} = \alpha$. Find the decrease in uncertainty $H(X) - H(X|Y)$. Apparently any set S with a given α is as good as any other.

(5)

42. **Inequalities.** Which of the following inequalities are generally $\geq, =, \leq$? Label each with $\geq, =$, or $\leq$.

   (a) $H(5X)$ vs. $H(X)$

   (b) $I(g(X); Y)$ vs. $I(X; Y)$

   (c) $H(X_0 | X_{-1})$ vs. $H(X_0 | X_{-1}, X_1)$

   (d) $H(X_1, X_2, \ldots, X_n)$ vs. $H(c(X_1, X_2, \ldots, X_n))$, where $c(x_1, x_2, \ldots, x_n)$ is the Huffman codeword assigned to $(x_1, x_2, \ldots, x_n)$.

   (e) $H(X, Y)/(H(X) + H(Y))$ vs. 1

(6)

43. **Mutual information of heads and tails.**

   (a) Consider a fair coin flip. What is the mutual information between the top side and the bottom side of the coin?

   (b) A 6-sided fair die is rolled. What is the mutual information between the top side and the front face (the side most facing you)?

(7)

7. **Entropy rates of Markov chains.**

   (a) Find the entropy rate of the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} 1 - p_{01} & p_{01} \\ p_{10} & 1 - p_{10} \end{bmatrix}.$$

   (b) What values of p_{01}, p_{10} maximize the rate of part (a)?

   (c) Find the entropy rate of the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} 1 - p & p \\ 1 & 0 \end{bmatrix}.$$

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