Probability of Error Decision

Page 8 of Topic 3

Objective: Find the error probability of ML decision rule.

The error probability based on the maximum likelihood detection on page 8 of Topic 3 is

$$\begin{aligned} P_e &= P\left[\left\{\{\text{“0” Tx} \cap \text{decide “1”}\}\right\} \cup \left\{\{\text{“1” Tx} \cap \text{decide “0”}\}\right\}\right] \\ &= P\left[\text{“0” Tx} \cap \text{decide “1”}\right] + P[\text{“1” Tx} \cap \text{decide “0”}] \\ &= P\left[\text{Decide } 1 \middle| \mathbf{s} = \mathbf{s}_0\right] P\left[\mathbf{s} = \mathbf{s}_0\right] + P\left[\text{Decide } 0 \middle| \mathbf{s} = \mathbf{s}_1\right] P\left[\mathbf{s} = \mathbf{s}_1\right]. \end{aligned}$$

Let's see first what $P\left[\text{Decide } 1 \middle| \mathbf{s} = \mathbf{s}_0\right]$ will be.

The receiver decides a "1" means that the receiver has a received signal $\mathbf{x}$ such that

$$\mathbf{x}^T\left(\mathbf{s}_0 - \mathbf{s}_1\right) < \frac{1}{2}\left(\left|\left|\mathbf{s}_0\right|\right|^2 - \left|\left|\mathbf{s}_1\right|\right|^2\right)$$

after correlation. Therefore, we have

$$\begin{aligned} P\left[\text{Decide } 1 \middle| \mathbf{s} = \mathbf{s}_0\right] &= P\left[\mathbf{x}^T\left(\mathbf{s}_0 - \mathbf{s}_1\right) < \frac{1}{2}\left(\left|\left|\mathbf{s}_0\right|\right|^2 - \left|\left|\mathbf{s}_1\right|\right|^2\right) \middle| \mathbf{s} = \mathbf{s}_0\right] \\ &= P\left[\left(\mathbf{s}_0 + \mathbf{n}\right)^T\left(\mathbf{s}_0 - \mathbf{s}_1\right) < \frac{1}{2}\left(\left|\left|\mathbf{s}_0\right|\right|^2 - \left|\left|\mathbf{s}_1\right|\right|^2\right)\right] \\ &= P\left[\mathbf{n}^T\left(\mathbf{s}_0 - \mathbf{s}_1\right) < \frac{-1}{2}\left|\left|\mathbf{s}_0 - \mathbf{s}_1\right|\right|^2\right], \end{aligned}$$

where $\mathbf{n}^T\left(\mathbf{s}_0 - \mathbf{s}_1\right)$ is a Gaussian random variable, since $\mathbf{n}$ is a jointly Gaussian random vector, with zero mean and variance

$$\text{Var}\left(\mathbf{n}^T\left(\mathbf{s}_0 - \mathbf{s}_1\right)\right) = E\left[(\mathbf{s}_0 - \mathbf{s}_1)^T\mathbf{n}\mathbf{n}^T(\mathbf{s}_0 - \mathbf{s}_1)\right] = \sigma^2\left|\left|\mathbf{s}_0 - \mathbf{s}_1\right|\right|^2.$$

Define $Y \triangleq \frac{\mathbf{n}^T(\mathbf{s}_0 - \mathbf{s}_1)}{\sigma||\mathbf{s}_0 - \mathbf{s}_1||}$. It is clear that Y is a standard normal random variable. Then, we have

$$P\left[\text{Decide } 1 \middle| \mathbf{s} = \mathbf{s}_0\right] = P\left[Y < \frac{-1}{2\sigma}\left|\left|\mathbf{s}_0 - \mathbf{s}_1\right|\right|\right] = Q\left(\frac{||\mathbf{s}_0 - \mathbf{s}_1||}{2\sigma}\right),$$

where the Q-function is

$$Q(x) \triangleq \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy.$$

Following a similar path, we can have

$$P\left[\text{Decide “1”} \middle| \mathbf{s} = \mathbf{s}_1\right] = Q\left(\frac{||\mathbf{s}_0 - \mathbf{s}_1||}{2\sigma}\right).$$

Therefore, the error probability is exactly

$$P_e = Q\left(\frac{||\mathbf{s}_0 - \mathbf{s}_1||}{2\sigma}\right),$$

from which we see the longer the distance $||\mathbf{s}_0 - \mathbf{s}_1||$ between $\mathbf{s}_0$ and $\mathbf{s}_1$, the smaller the error probability. This is an intuitively correct result.