Stochastic Processes

Final Exam

9:00 a.m. – 11:30 a.m., 01/12/2009

IMPORTANT:

- Remember to write down your id number and your name.
- There are 5 problem sets with 120 points in total.
- Please provide detailed explanations/reasonings with your answers. Correct answers ***without*** any explanations will carry **NO** credits. On the other hand, wrong answers with correct reasonings will get partial credits.

You may need the following formulas:

- The probability mass function for a Poisson process $N(t)$ at time t is given by

$$P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t} u(t),$$

where λ is the rate of the process, $u(t)$ is the unit step function and n is a non-negative integer.

- The probability density function of an exponential random variable X with rate λ is given by

$$f_X(x) = \lambda e^{-\lambda x} u(x),$$

where $u(x)$ is the unit step function. The expected value of X is $\frac{1}{\lambda}$.

1. ($8\times$ 5=40 points)

   Answer the following questions. Please be ***clear*** and ***rigorous***.

   (a) For random vectors $\mathbf{x}$ and $\mathbf{y}$ with ***arbitrary*** distributions, the orthogonality principle states that $\mathbf{x} - E[\mathbf{x}|\mathbf{y}]$ is statistically orthogonal to $k(\mathbf{y})$ for any function $k(\cdot)$. Justify this statement.

   (b) What is the definition of convergence in probability for a random sequence? And, what is the definition of convergence with probability one? Explain the difference between them.

   (c) Clearly state what the Karhunen-Loéve expansion of a random process is. Also clearly state its counterpart in the case of a random vector.

   (d) Show that any discrete-valued independent-increment random process is also a Markov process.

   (e) Clearly state the Wiener-Kinchine theorem. And, explain why and under what conditions can we say $E[|X[n]|^2]$ is the ***average power*** of the random sequence $X[n]$.

2. **Random Sequence in LTI Systems** (5+5+7=17 points)

Consider a linear time-invariant system with impulse response $h[n]$. We input a WSS random sequence $W[n]$ with zero mean and autocorrelation function $R_{WW}[m] = \delta[m]$ into the system, and get an output random sequence $X[n]$.

(a) Show that $X[n]$ is also WSS with autocorrelation function

$$R_{XX}[m] = h[m] \star h^*[-m],$$

where $\star$ is the convolution operation.

(b) Suppose the system output $X[n]$ has the following recursive relation

$$X[n] = aX[n-1] + bW[n]. \tag{1}$$

Find the system impulse response $h[n]$.

(c) Typically, a zero mean random sequence $X[n]$ with a known autocorrelation function can be simulated numerically using the recursive relation in equation (1). Suppose we want $X[n]$ in part (b) to be zero mean with average power σ^2 and nearest neighbor correlation $R_{XX}[1] = \rho\sigma^2$. How should we design a and b (in terms of ρ and σ^2) to achieve this goal?

3. **Bernoulli Random Sequence** (7×5=35 points)

Consider a sequence $X[n]$ of Bernoulli trials, where each trial produces a 1 (a success) with probability p, and a 0 (a failure) with probability $1-p$, independently of what happens in other trials. Let $Y[k]$ be the positive integer that records the number of trials needed until the kth success in the Bernoulli random sequence has arrived. Assume $Y[0]=0$.

(a) Find the mean and autocorrelation function of $X[n]$.

(b) Are $X[n]$ stationary in strict sense? Justify your answer.

(c) What is the probability of exactly S successes recoded in the first N trials?

(d) The inter-arrival time $T[k]$ between the $(k-1)$th and the kth success is defined to be

$$T[k] = Y[k] - Y[k-1].$$

Find, at the time instant k, the probability mass function of $T[k]$.

(e) Determine, at the time instant k, the probability mass function of $Y[k]$. (Hint: $Y[k] = \sum_{i=1}^{k} T[k]$)

4. **WSS $\nrightarrow$ Stationary** ($7 + 7 = 14$ points)

Consider a random process

$$X(t) = A \cdot \cos(2\pi f t) + B \cdot \sin(2\pi f t),$$

where A and B are i.i.d. random variables with zero mean, variance σ^2, and nonzero third moment $E[A^3] = E[B^3] \neq 0$. The frequency f is a constant.

(a) Show that the random process $X(t)$ is WSS.

(b) Show that $X(t)$ is not strictly stationary.

5. **Poisson Process** (7+7=14 points)

Suppose the number of fish that a fisherman catches follows a Poisson process with rate $\lambda = 0.5$ per hour. The fisherman will keep fishing for two hours. If he has caught at least one fish, he quits. Otherwise, he continues until he catches at least one fish.

(a) Find the probability that he catches at least three fish.

(b) Find the expected total fishing time, given that he has been fishing for four hours.