Stochastic Processes

Final Exam

Solutions

1. (8× 5=40 points)

   Answer the following questions. Please be ***clear*** and ***rigorous***.

   (a) See the topic 5 lecture note.

   (b) **Definition (Convergence with Probability 1)**
   The random sequence $X[n]$ converges ***almost surely*** (with probability one) to the random variable X if the sequence $X[n, \varepsilon]$ converges $X[\varepsilon]$ for all $\varepsilon \in \Omega$ except possibly on a set of probability 0.
   Definition (Convergence in Probability)
   A random sequence $X[n]$ converges in probability to the random variable X if for every $\varepsilon > 0$,

   $$\lim_{n\to\infty} P[|X[n] - X| > \varepsilon] = 0.$$

   The difference between them is well explained on page 380 in the textbook.

   (c) See lecture note in topic 8.

   (d) See textbook p. 423.

   (e) For a ***zero mean WSS*** random sequence $X[n]$, the average power is obtained by integrating the power spectral density with respect to the entire frequency band

   $$\begin{aligned} P_{\text{ave}} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega m}d\omega\Big|_{m=0} \\ &= R_{XX}[0] \\ &= E[|X[n]|^2]. \end{aligned}$$

2. **Random Sequence in LTI Systems** (5+5+7=17 points)

   (a) We know the autocorrelation function of $X[n]$ is

   $$R_{XX}[m] = h[m] \star h^*[-m] \star R_{WW}[m],$$

   where $\star$ is the convolution operation. But we know $R_{WW}[m] = \delta[m]$, hence

   $$R_{XX}[m] = h[m] \star h^*[-m].$$

   (b) The impulse response $h[n]$ is the output when the input is an impulse. That is

   $$h[n] = ah[n-1] + b\delta[n],$$

   which leads to

   $$h[n] = a^n bu[n],$$

   where $u[n]$ is the unit step function.

   (c) See page 357 in the textbook, we need $a = \rho$ and $b^2 = \sigma^2(1-\rho^2)$.

3. **Bernoulli Random Sequence** (7×5=35 points)

(a) The mean function is

$$E[X[n]] = p$$

and the autocorrelation function is

$$\begin{aligned} R_{XX}[m,n] &= E[X[m]X[n]] \\ &= \begin{cases} p & m = n \\ p^2 & m \neq n \end{cases} \\ &= p^2 + p(1-p)\delta[m-n]. \end{aligned}$$

(b) Yes, it is stationary, it is relatively straightforward to show the joint PMF of any order is shift-invariant due to the i.i.d. property of the sequence.

(c) The number of success in N trials is just a binomial random variable. Thus,

$$P[\text{number} = s] = \binom{N}{S} p^S (1-p)^{N-S}.$$

(d) The inter-arrival time $T[k]$ between the $(k-1)$th and the kth success is defined to be

$$T[k] = Y[k] - Y[k-1].$$

Find the mean, variance, and probability mass function of $T[k]$.
The event of $T[k] = t$ means there are exactly $t-1$ consecutive zeros (failures) until the kth success has been recorded. Therefore, we have

$$\begin{aligned} P[T[k] = t] &= P\Big[X[Y[k]] = 1 \cap X[Y[k]-1] = 0 \cap X[Y[k]-2] = 0 \cdots \cap X[Y[k]-t+1] = 0\Big] \\ &= (1-p)^{t-1} p. \end{aligned}$$

We can obtain the mean and variance of $T[k]$ from the PMF

$$E[T[k]] = 1/p \quad \text{and} \quad \text{Var}(T[k]) = \frac{1-p}{p^2}$$

(e) Since $Y[k] = \sum_{i=1}^{k} T[i]$, we have

$$E[Y[k]] = \sum_{i=1}^{k} E[T[i]] = \frac{k}{p}$$

and

$$\begin{aligned} \text{Var}(Y[k]) &= \text{Var}\left(\sum_{i=1}^{k} Y[i]\right) \\ &= \sum_{i=1}^{k} \text{Var}(Y[i]) = k\frac{1-p}{p^2}, \end{aligned}$$

where we have used that fact that $T[k]$ is an independent sequence.
The event of $Y[k] = y$ means the time of the kth success is y. In other words, up to and including time $y-1$, we have already recorded $k-1$ successes, which is exactly what we have done in Part (b). Consequently, when incorporating the kth success, the PMF of $Y[k]$ is

$$P[Y[k] = y] = \binom{y-1}{k-1} p^k (1-p)^{y-k}, \quad y = k, k+1, \cdots.$$

4. **WSS $\nrightarrow$ Stationary** $(7 + 7 = 14$ points)

See solutions for Prob. 3 of homework 5.

5. **Poisson Process** (7+7=14 points)

Let N_t denote the Poisson process of the problem.

(a) If he catches at least three fish, he must have fished for exactly two hours. Hence, the desired probability is equal to the probability that the number of fish caught in the first two hours is at least three,

$$\begin{aligned} P(N_2 \geq 3) &= 1 - P(N_2 = 0) - P(N_2 = 1) - P(N_2 = 2) \\ &= 1 - e^{-1} - e^{-1} - \frac{1}{2}e^{-1} \\ &= 1 - \frac{5}{2}e^{-1}. \end{aligned}$$

(b) Given that he has been fishing for 4 hours, the future fishing time is the time until the first fish is caught. By the memoryless property of the Poisson process, the future time is exponential, with mean $1/\lambda$. Hence, the expected total fishing time is $4 + (1/0.5) = 6$.