Homework 1

Due on Tuesday, **10/6/2009**, in class

1. ($6 + 7 + 7 = 20$ points) Alice and Bob work independently on a problem set. The time for Alice to complete the set is exponentially distributed with mean 4 hours. The time for Bob to complete the set is exponentially distributed with mean 6 hours.

   (a) What is the probability that Alice finishes the problem set before Bob?

   (b) Given that Alice requires more than 4 hours, what is the probability that she finishes the problem set before Bob?

   (c) What is the probability that one of them finishes the problem set an hour or more before the other one?

2. ($10 + 10 = 20$ points) Peter continually and *independently* flips a coin having probability p of coming up heads. He decides to stop flipping when the 2nd head appears. Let X denote the number of flips needed for the 1st head to appear, and Y be the additional number of flips for the 2nd head to show up after the 1st head has appeared.

   (a) Find the probability mass function (pmf) of Y and the pmf of $X + Y$.

   (b) Find $P(X = k|X + Y = n)$ and $E[X|X + Y = n]$.

3. ($5 + 10 + 10 = 25$ points) Prove the followings.

   (a) For any random variables X and Y, show that $E[E[X|Y]] = E[X]$.

   (b) Let X and Y be two random variables, either discrete or continuous. Show that

   $$E\Big[g(X) \cdot h(Y)\Big|Y = y\Big] = h(y) \cdot E\Big[g(X)\Big|Y = y\Big]$$

   for any real-valued functions g and h. (Therefore, we can write

   $$E\Big[g(X) \cdot h(Y)\Big|Y\Big] = h(Y) \cdot E\Big[g(X)\Big|Y\Big]$$

   to emphasize the whole thing is a random variable as a function of Y.)

   (c) Show that, for any function $k(\cdot)$, we have

   $$E\big[\big(X - E[X|Y]\big) \cdot k(Y)\big] = 0.$$

4. ($5 + 10 + 10 = 25$ points) In class, I talked about the singular value decomposition (SVD) to factor any $m \times n$ rectangular matrix $\mathbf{A}$ with rank r into

   $$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^H,$$

   where $\mathbf{U} = [\mathbf{U_1}|\mathbf{U_2}]$ and $\mathbf{V} = [\mathbf{V_1}|\mathbf{V_2}]$ are $m \times m$ and $n \times n$ unitary matrices, respectively, and

   $$\mathbf{D} = \left[\begin{array}{c|c} \mathbf{\Sigma}_{r\times r} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]$$

   is a simple structured $m \times n$ matrix with $\mathbf{\Sigma}_{r\times r} = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$. You can see that the singular values $\sigma_1, \cdots, \sigma_r$ are actually the square roots of the eigenvalues of $\mathbf{A}^H\mathbf{A}$. In this problem, we will look further into the structure of the matrices $\mathbf{U}$ and $\mathbf{V}$.

   (a) Show that the singular values are also the square roots of the eigenvalues of $\mathbf{A}\mathbf{A}^H$.

   (b) Show that $\mathrm{rank}(\mathbf{A}^H\mathbf{A}) = \mathrm{rank}(\mathbf{A})$. This is why the matrix $\mathbf{A}^H\mathbf{A}$ has r non-zero eigenvalues.

   (c) Show that the range of $\mathbf{U_1}$ is the range of $\mathbf{A}$, and the range of $\mathbf{V_2}$ is the null space of $\mathbf{A}$.

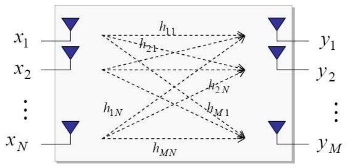
5. (5+5=10 points) Multiple-input multiple-output (MIMO) systems have been shown to offer significant increase in the channel capacity and for wireless communication systems. The basic idea is to implement multiple antennas at both transmitter and receiver end. Typically, we can model an $M \times N$ MIMO system with N transmit and M receive antennas as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n},$$

where $\mathbf{x} = [x_1\ x_2\ \cdots\ x_N]^T$, $\quad \mathbf{y} = [y_1\ y_2\ \cdots\ y_M]^T$, $\quad \mathbf{n} = [n_1\ n_2\ \cdots\ n_M]^T$ and

$$\mathbf{H} = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,N} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{M,1} & h_{M,2} & \cdots & h_{M,N} \end{pmatrix}.$$

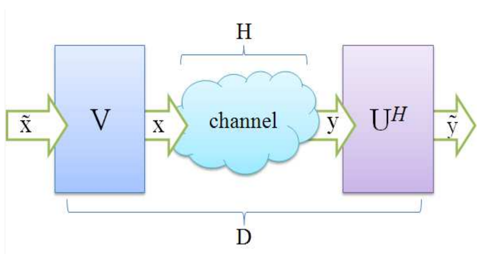
More specifically, x_j is the transmitted symbol at the jth antenna of the transmitter ($j = 1,\ldots,N$), y_i is the received symbol at the ith antenna of the receiver ($i = 1,\ldots,M$), $\mathbf{n}$ denotes the noise vector at receiver ($M \times 1$) and $h_{i,j}$ is the channel gain from the jth transmit antenna to the ith receive antenna. The system structure is shown in the figure below.



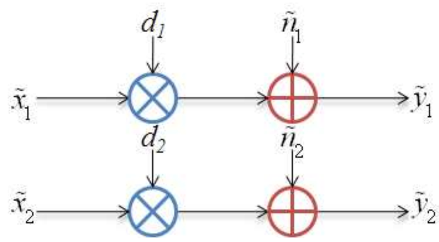
The MIMO channels can be decomposed into many parallel subchannels using a precoder technique. That is, multiplying $\mathbf{x}$ by a matrix $\mathbf{T}$ before transmission and multiplying $\mathbf{y}$ by a matrix $\mathbf{R}$ after receiving. To simplify this problem, let's consider a 2×2 MIMO system with the channel matrix $\mathbf{H}$

$$\mathbf{H} = \begin{pmatrix} 1 & \sqrt{3} \\ 2 & 0 \end{pmatrix}.$$

(a) Design $\mathbf{V}$ and $\mathbf{U}$ in Fig. (a) such that $\tilde{\mathbf{y}} = \mathbf{U}^H\mathbf{H}\mathbf{V}\tilde{\mathbf{x}} + \mathbf{U}^H\mathbf{n} = \mathbf{D}\tilde{\mathbf{x}} + \tilde{\mathbf{n}}$, where $\mathbf{D}$ is a diagonal matrix. With this, we obtain two parallel subchannels as shown in Fig. (b).



(a)



(b)

(b) Now, assume that the total transmit power is fixed, i.e., (power on $\tilde{x}_1$) + (power on $\tilde{x}_2$) = const. Which one would you like to give more power on? $\tilde{x}_1$ or $\tilde{x}_2$? And why?
(Hint: the answer may depend on how you do with part (a))
(Actually, there's an optimum power allocation strategy to maximize the "MIMO capacity," but we don't discuss that here.)