Homework 2

Due on **10/20/2009, 3:40 pm**, in class before midterm exam

Part I: Reading Assignment

1. Read textbook Sec. 3.2 $\sim$ Sec. 3.5, Sec. 4.1, and Sec. 4.2, which you should have learned before in college level Probability course.
2. Read Chapter 2 of Gallager's note.

Part II: Problem Assignment

1. ($6 \times 5 = 30$ points)
   Let $\mathbf{w} = [X, Y, Z]^T \sim \mathcal{N}(\mathbf{m}_{\mathsf{w}}, \mathbf{K}_{\mathsf{w}})$, where $\mathbf{m}_{\mathsf{w}} = [1\ 2\ 0]^T$ and the covariance matrix $\mathbf{K}_{\mathsf{w}}$ is

$$\mathbf{K}_{\mathsf{w}} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

   (a) Find the joint moment generating function of $\mathbf{w}$.
   (b) Show that $X - \alpha Y$ and Y are jointly Gaussian for any real α. Then, find α such that $X - \alpha Y$ and Y are independent.
   (c) What is the conditional expectation $E[Y^3|Z]$?
   (d) Find the probability density function (pdf) for $S = X + 2Y$.
   (e) Let $\mathbf{s} = [X, Y]^T$. Find a transformation matrix $\mathbf{T}$ such that $\mathbf{Ts}$ is a Gaussian random vector with uncorrelated components of unit variance.
2. ($6 + 7 + 7 = 20$ points)
   In a common communication model, the transmitted signal $\mathbf{x}$ and the received signal $\mathbf{y}$ are related by:

$$\mathbf{y} = \mathbf{Hx} + \mathbf{z},$$

   where $\mathbf{H}$ is an n by m matrix that is not random, $\mathbf{x}$ is jointly Gaussian with $\mathcal{N}(0; \mathbf{K}_x)$ and $\mathbf{z}$ is additive Gaussian noise $\mathcal{N}(0; \mathbf{K}_z)$. The noise is independent of the signal $\mathbf{x}$.
   (a) Let $\mathbf{u} = [\mathbf{x}^T; \mathbf{y}^T]^T$. Show that $\mathbf{u}$ is jointly Gaussian.
   (b) Find a simple condition on $\mathbf{H}$, $\mathbf{K}_x$ and $\mathbf{K}_z$ for which the covariance matrix $\mathbf{K}_u$ of $\mathbf{u}$ is invertible.
   (c) Find the conditional distribution of the input $\mathbf{x}$ given the output $\mathbf{y} = \boldsymbol{y}$.
3. (10 points)
   Show that a circularly symmetric complex Gaussian random variable must have i.i.d. real and imaginary components.
4. ($10 + 10 + 10 = 30$ points)
   Problem 2.13 in Gallager's note.
5. (10 points)
   Problem 4.19 in textbook.

`Extra Problems`

You do <u>`NOT`</u> need to turn in your solutions for the following problems. However, I strongly encourage you to work through them.

1. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be collectively jointly Gaussian random vectors. That is the components of the random vector $[\mathbf{x}^T, \mathbf{y}^T, \mathbf{z}^T]^T$ are jointly Gaussian.

   (a) If $\mathbf{y}$ and $\mathbf{z}$ are statistically independent, show that

   $$E[\mathbf{x}|\mathbf{y}, \mathbf{z}] = E[\mathbf{x}|\mathbf{y}] + E[\mathbf{x}|\mathbf{z}] - \mathbf{m}_{\mathbf{x}},$$

   where $\mathbf{m}_{\mathbf{x}} = E[\mathbf{x}]$.

   Hint:
   Let $\mathbf{s} \triangleq [\mathbf{y}^T \mathbf{z}^T]^T$ be the $(m+r) \times 1$ vector collecting $\mathbf{y}$ and $\mathbf{z}$. And, $E[\mathbf{x}|\mathbf{y}, \mathbf{z}] = E[\mathbf{x}|\mathbf{s}]$.

   (b) If $\mathbf{y}$ and $\mathbf{z}$ are not necessarily statistically independent, show that

   $$E[\mathbf{x}|\mathbf{y}, \mathbf{z}] = E[\mathbf{x}|\mathbf{y}, \hat{\mathbf{z}}],$$

   where $\hat{\mathbf{z}} = \mathbf{z} - E[\mathbf{z}|\mathbf{y}]$.