

Homework 3

Due on 11/17/2009, Tuesday, *before* class

Reading Assignments

Chapter 3 and Chapter 4 of Gallager's note. Sec. 4.8, Sec. 5.8, Sec. 5.9, and Sec. 5.10 of textbook.

Problem Assignments

1. (10 points) Show that the Q -function, defined by $Q(x) = P[Z \geq x] = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}} dz$ with standard Gaussian random variable Z , has the following upper bound for $x \geq 0$:

$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}.$$

2. (10+10=20 points) Consider a binary hypothesis testing problem, and denote the hypotheses as $H_1 : \alpha = 1$ and $H_2 : \alpha = -1$. Let \mathbf{h} be an $n \times 1$ real vector and the observation random vector \mathbf{y} takes the form

$$\mathbf{y} = \alpha \cdot \mathbf{h} + \mathbf{z},$$

where $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ is independent with α and \mathbf{I}_n is an $n \times n$ identity matrix.

- (a) Find the maximum likelihood decision rule for α and find the probability of error in terms of the Q function.
- (b) Now suppose a third hypothesis $\alpha = 0$ is added to the situation of part (a). The observation random vector takes the same form, but here α can take on values $-1, 0$ or 1 , equally likely. Find the maximum likelihood decision rule in this case, and determine the error probability.
3. ($10 \times 4 = 40$ points) Let the mathematical model of a communication system with N transmit and M receive antennas be given by

$$\mathbf{y}_i = \sqrt{\frac{E_s}{N}} \mathbf{H} \mathbf{s}_i + \mathbf{w}_i,$$

where \mathbf{y}_i is the received signal at the i th time instant with dimension $M \times 1$, \mathbf{H} is the time-invariant channel matrix with dimension $M \times N$, \mathbf{s}_i is the transmitted signal vector (with unit average energy) at the i th time instant, E_s is the total energy of the transmitted signal, and \mathbf{w}_i is a circularly symmetric complex Gaussian random vector with $\mathbf{w}_i \sim \mathcal{CN}(\mathbf{0}, N_0 \mathbf{I}_M)$. Suppose we collect T signal vectors, from $i = 1$ to $i = T$, and stack them into

$$\mathbf{Y} = \sqrt{\frac{E_s}{N}} \mathbf{H} \mathbf{S} + \mathbf{W},$$

where $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_T]$, and likewise for the representations of \mathbf{S} and \mathbf{W} . The signal codeword matrix \mathbf{S} belongs to a set $\mathcal{S} = \{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \dots, \mathbf{S}^{(K)}\}$, called codebook, with size K .

- (a) Given that the receiver knows the channel matrix \mathbf{H} , show that the ML detection rules gives the following decision criterion:

$$\hat{\mathbf{S}}_{ML} = \arg \min_{\mathbf{S}} \|\mathbf{Y} - \mathbf{H} \mathbf{S}\|_F^2,$$

where the Frobenious norm $\|\mathbf{A}\|_F$ of a $p \times q$ matrix \mathbf{A} is defined as $\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^p \sum_{j=1}^q |a_{i,j}|^2}$, where $a_{i,j}$ is the (i,j) th entry of the matrix \mathbf{A} .

- (b) With the ML decision rule in part (a), show that the conditional pairwise error probability $P(\mathbf{S}^{(i)} \rightarrow \mathbf{S}^{(j)} | \mathbf{H})$, which means the probability that the receiver decides the codeword $\mathbf{S}^{(j)}$ while the actual transmitted codeword is $\mathbf{S}^{(i)}$, is given by

$$P(\mathbf{S}^{(i)} \rightarrow \mathbf{S}^{(j)} | \mathbf{H}) = Q\left(\sqrt{\frac{\rho \|\mathbf{H}\mathbf{E}_{i,j}\|_F^2}{2N}}\right),$$

where $\rho = \frac{E_s}{N_0}$ is the signal-to-noise ratio (SNR) and $\mathbf{E}_{i,j} = \mathbf{S}^{(i)} - \mathbf{S}^{(j)}$.

(Hint: You can use $\|\mathbf{A}\|_F^2 = \text{Tr}(\mathbf{A}^H \mathbf{A})$, the trace of $\mathbf{A}^H \mathbf{A}$, to simplify the derivation.)

- (c) From Problem 3 and part (b) of this problem, the conditional pairwise probability can be bounded by

$$P(\mathbf{S}^{(i)} \rightarrow \mathbf{S}^{(j)} | \mathbf{H}) \leq e^{-\frac{\rho \|\mathbf{H}\mathbf{E}_{i,j}\|_F^2}{4N}}.$$

Show that the average PEP can be bounded by

$$P(\mathbf{S}^{(i)} \rightarrow \mathbf{S}^{(j)}) \leq \left(\frac{1}{\det\left(\mathbf{I}_N + \frac{\rho}{4N} \mathbf{E}_{i,j} \mathbf{E}_{i,j}^H\right)} \right)^M,$$

where the average is taken over the channel matrix \mathbf{H} whose components are assumed to be i.i.d. complex Gaussian random variables with zero mean and unit variance, *i.e.* the (i, j) th component $h_{i,j} \sim \mathcal{CN}(0, 1)$.

- (d) Discuss how the average PEP in part (c) scales with the number of transmit (N) and receive (M) antennas as the SNR goes to infinity.
4. (10+10=20 points) A gamma random variable Y_k with parameter (k, λ) can be considered as the sum of k independent exponential random variables T_i for $i = 1 \cdots k$ with parameter λ , *i.e.*

$$Y_k = \sum_{i=1}^k T_i.$$

The gamma random variable Y_k is commonly used to model the total **amount of time** that one has to wait until the k th Poisson event has occurred. Or, equivalently, T_i is used to model the inter-arrival time between the $(i-1)$ th and the i th occurrence of the Poisson event, where the time of the zero-th occurrence is labelled as the origin.

- (a) Find the probability distribution function $F_{Y_k}(y) = P[Y_k \leq y]$ of Y_k .
(Hint: consider the event that the number of Poisson events occurred in the interval $[0, y]$ is greater than k .)
- (b) Find the probability density function of Y_k by differentiating $F_{Y_k}(t)$ in part (a).
5. (10 points) Justify that the sample variance is an unbiased estimator.

Extra Problems

You do NOT need to turn in your solutions for the following problems. However, I strongly encourage you to work through them.

1. Consider a communication system that is corrupted by unknown interference Z and Gaussian noise W . Mathematically, the received signal Y can be modeled as

$$Y = X + Z + W,$$

where X is the desired signal equally likely to 1 or -1 , and $W \sim \mathcal{N}(0, \sigma^2)$. Assume X , Z , W are mutually independent.

- (a) Suppose we model the interference Z as a Gaussian random variable with $Z \sim \mathcal{N}(0, A^2)$. Find the maximum likelihood decision rule for X from the received signal Y . What is the probability of error decision? And, how does the error probability behave when the power A^2 approaches to infinity?
- (b) More practically, we now model the interference Z as a binary discrete random variable equally likely to be A or $-A$. Find the maximum likelihood decision rule for X from the received signal Y .
- (c) In part (b), obtain the ML decision rule when A approaches to infinity. And, calculate the probability of error decision in this large power condition.
- (d) Explain intuitively the difference between part (a) and part (c).