Homework 3

Due on **11/17/2009**, Tuesday, *before* class

Reading Assignments

Chapter 3 and Chapter 4 of Gallager's note. Sec. 4.8, Sec. 5.8, Sec. 5.9, and Sec. 5.10 of textbook.

Problem Assignments

1. (10 points) Show that the Q-function, defined by $Q(x) = P[Z \geq x] = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{z^2}{2}} dz$ with standard Gaussian random variable Z, has the following upper bound for $x \geq 0$.:
$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}.$$

2. (10+10=20 points) Consider a binary hypothesis testing problem, and denote the hypotheses as $\mathrm{H}_1: \ \alpha = 1$ and $\mathrm{H}_2: \ \alpha = -1$. Let $\mathbf{h}$ be an $n \times 1$ real vector and the observation random vector y takes the form
$$\mathsf{y} = \alpha \cdot \mathbf{h} + \mathsf{z},$$
where $\mathsf{z} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ is independent with α and $\mathbf{I}_n$ is an $n \times n$ identity matrix.
   (a) Find the maximum likelihood decision rule for α and find the probability of error in terms of the Q function.
   (b) Now suppose a third hypothesis $\alpha = 0$ is added to the situation of part (a). The observation random vector takes the same form, but here α can take on values $-1, 0$ or 1, equally likely. Find the maximum likelihood decision rule in this case, and determine the error probability.

3. ($10 \times 4 = 40$ points) Let the mathematical model of a communication system with N transmit and M receive antennas be given by
$$\mathbf{y}_i = \sqrt{\frac{E_s}{N}} \mathbf{H} \mathbf{s}_i + \mathbf{w}_i,$$
where $\mathbf{y}_i = [y_{i,1}, y_{i,2}, \cdots, y_{i,M}]^T$ is the $M \times 1$ vector collecting received signals from all receive antennas at the ith time instant, $\mathbf{H}$ is the time-invariant channel matrix with dimension $M \times N$, $\mathbf{s}_i$ is the $N \times 1$ transmitted signal vector (with unit average energy) at the ith time instant, E_s is the total energy of the transmitted signal, and $\mathbf{w}_i \sim \mathcal{CN}(\mathbf{0}, \mathtt{N}_0 \mathbf{I}_M)$ is the complex Gaussian noise vector with $\mathtt{N}_0$ being the variance of each noise component and $\mathbf{I}_M$ the identity matrix with size $M \times M$. More specifically, the model tells that the mth component $y_{i,m}$ of $\mathbf{y}_i$ represents the received signal at the mth antenna of the receiving device, and is equal to
$$y_{i,m} = \sum_{n=1}^{N} h_{m,n} s_{i,n} + w_{i,m},$$
where $h_{m,n}$ is the (m,n)th component of $\mathbf{H}$ which means the channel gain between the nth transmit antenna and the mth receive antenna, $s_{i,n}$ is the nth component of $\mathbf{s}_i$ signifying the transmitted signal from the nth transmit antenna at the ith time instant, and $w_{i,m}$ is the noise.

We collect T received signal vectors, from $i = 1$ to $i = T$, and stack them into

$$\mathbf{Y} = \sqrt{\frac{E_s}{N}}\mathbf{HS} + \mathbf{W},$$

where $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_T]$, and likewise for the representations of $\mathbf{S}$ and $\mathbf{W}$. The signal codeword matrix $\mathbf{S}$ belongs to a set $\mathcal{S} = \{\mathbf{S}^{(1)}, \mathbf{S}^{(2)}, \cdots, \mathbf{S}^{(K)}\}$, called codebook, with size K.

(a) Given that the receiver knows the channel matrix $\mathbf{H}$, show that the ML detection rules gives the following decision criterion:

$$\hat{\mathbf{S}}_{ML} = \arg\min_{\mathbf{S}} \left|\left|\mathbf{Y} - \sqrt{\frac{E_s}{N}}\mathbf{HS}\right|\right|_F^2,$$

where the Frobenious norm $||\mathbf{A}||_F$ of a $p \times q$ matrix $\mathbf{A}$ is defined as the sum of squared absolute values of all components, *i.e.* $||\mathbf{A}||_F = \sqrt{\sum_{i=1}^{p}\sum_{j=1}^{q}|a_{i,j}|^2}$, where $a_{i,j}$ is the (i,j)th entry of the matrix $\mathbf{A}$.

(b) With the ML decision rule in part (a), show that the conditional pairwise error probability $P\left(\mathbf{S}^{(i)} \to \mathbf{S}^{(j)}|\mathbf{H}\right)$, which means the probability that the receiver decides the codeword $\mathbf{S}^{(j)}$ while the actual transmitted codeword is $\mathbf{S}^{(i)}$, is given by

$$P\left(\mathbf{S}^{(i)} \to \mathbf{S}^{(j)}|\mathbf{H}\right) = Q\left(\sqrt{\frac{\rho||\mathbf{HE}_{i,j}||_F^2}{2N}}\right),$$

where $\rho = \frac{E_s}{\mathrm{N}_0}$ is the signal-to-noise ratio (SNR) and $\mathbf{E}_{i,j} = \mathbf{S}^{(i)} - \mathbf{S}^{(j)}$. (Hint: You can use $||\mathbf{A}||_F^2 = \mathrm{Tr}(\mathbf{A}^H\mathbf{A})$, the trace of $\mathbf{A}^H\mathbf{A}$, to simplify the derivation.)

(c) From Problem 3 and part (b) of this problem, the conditional pairwise probability can be bounded by

$$P\left(\mathbf{S}^{(i)} \to \mathbf{S}^{(j)}|\mathbf{H}\right) \leq e^{-\frac{\rho||\mathbf{HE}_{i,j}||_F^2}{4N}}.$$

Show that the average PEP can be bounded by

$$P\left(\mathbf{S}^{(i)} \to \mathbf{S}^{(j)}\right) \leq \left(\frac{1}{\det\left(\mathbf{I}_N + \frac{\rho}{4N}\mathbf{E}_{i,j}\mathbf{E}_{i,j}^H\right)}\right)^M,$$

where the average is taken over the channel matrix $\mathbf{H}$ whose components are assumed to be i.i.d. complex Gaussian random variables with zero mean and unit variance, *i.e.* the (i,j)th component $h_{i,j} \sim \mathcal{CN}(0,1)$.

(d) Discuss how the average PEP in part (c) scales with the number of transmit (N) and receive (M) antennas as the SNR goes to infinity.

4. (10+10=20 points) A gamma random variable Y_k with parameter (k, λ) can be considered as the sum of k independent exponential random variables T_i for $i = 1 \cdots k$ with parameter λ, *i.e.*

$$Y_k = \sum_{i=1}^{k} T_i.$$

The gamma random variable Y_k is commonly used to model the total ***amount of time*** that one has to wait until the kth Poisson event has occurred. Or, equivalently, T_i is used to model the inter-arrival time between the $(i-1)$th and the ith occurrence of the Poisson event, where the time of the zero-th occurrence is labelled as the origin.

(a) Find the probability distribution function $F_{Y_k}(y) = P\big[Y_k \le y\big]$ of Y_k. (Hint: consider the event that the number of Poisson events occurred in the interval $[0, y]$ is greater than k.)

(b) Find the probability density function of Y_k by differentiating $F_{Y_k}(t)$ in part (a).

5. (10 points) Justify that the sample variance is an unbiased estimator.

`Extra Problems`

You do <u>`NOT`</u> need to turn in your solutions for the following problems. However, I strongly encourage you to work through them.

1. Consider a communication system that is corrupted by unknown interference Z and Gaussian noise W. Mathematically, the received signal Y can be modeled as

$$Y = X + Z + W,$$

where X is the desired signal equally likely to 1 or -1, and $W \sim \mathcal{N}(0, \sigma^2)$. Assume X, Z, W are mutually independent.

(a) Suppose we model the interference Z as a Gaussian random variable with $Z \sim \mathcal{N}(0, A^2)$. Find the maximum likelihood decision rule for X from the received signal Y. What is the probability of error decision? And, how does the error probability behave when the power A^2 approaches to infinity?

(b) More practically, we now model the interference Z as a binary discrete random variable equally likely to be A or $-A$. Find the maximum likelihood decision rule for X from the received signal Y.

(c) In part (b), obtain the ML decision rule when A approaches to infinity. And, calculate the probability of error decision in this large power condition.

(d) Explain intuitively the difference between part (a) and part (c).