Homework 4

Due on **12/29/2009**, Tuesday, before class

Reading assignments:

1. Sec. 6.1 ~ Sec. 6.7, textbook.
2. Sec. 7.1~ Sec. 7.6, textbook.

Problem assignments:

1. A gambler plays a sequence of independent games. In each game, he wins s dollars with probability p and loses s dollars with probability $q = 1 - p$, where s is a positive real number, typically integer. Let $X[n]$ represent the amount of money the gambler has after n games, with $X[0]$ being the initial amount of money the gambler has prior to the first game. Mathematically, the random sequence $X[n]$ can be modeled as the running sum

$$X[n] \triangleq X[0] + \sum_{i=1}^{n} W[i], \quad n \geq 1,$$

where

$$W[i] = \begin{cases} +s, & \text{with } P\big[W[i] = s\big] = p, \\ -s, & \text{with } P\big[W[i] = -s\big] = q. \end{cases}$$

The sequence is exactly a ***random walk*** starting at position $X[0]$.

(a) Let $X[0] = 0$. Find the probability $P\big[X[n] = rs\big]$ that after n games the gambler has rs dollars for some integer r. (Note that $r < 0$ is allowed here, meaning the gambler is in debt.)

(b) Let $X[0] = 0$. Find the mean function $E\big[X[n]\big]$, the variance function $\mathrm{Var}\big(X[n]\big)$, and autocorrelation function $R_{XX}[m, n] = E\big[X[m]X^*[n]\big]$ for $m, n > 0$.

(c) Now suppose the gambler initially has $X[0] = K$ dollars and $s = 1$. He continues playing the games until he either accumulates N dollars (including his initial K dollars) or has no money left, where $N > K$. Find the probability P_K that the gambler will end up with N dollars.
(Hint: Condition on the outcome of the first game and use total probability.)

(d) Following the assumptions in part (c). Show that the probability Q_K that the gambler will end up having no money left is

$$Q_K = \begin{cases} \frac{1-(p/q)^{N-K}}{1-(p/q)^N} & \text{if } q \neq \frac{1}{2}, \\ \frac{N-K}{N} & \text{if } q = \frac{1}{2}. \end{cases}$$

(e) Give physical interpretations to the gambler's ruin probability Q_K in part (d).

2. Problem 6.23 in textbook.
3. Problem 6.24 in textbook.
4. Problem 6.36 in textbook.
5. Problem 7.8 in textbook.
6. Problem 7.17 in textbook.
7. Problem 7.34 in textbook.

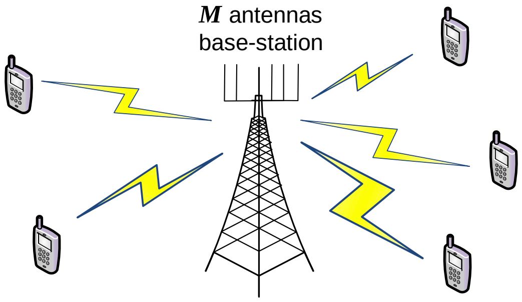

Figure 1: Illustration of Extra Problem 1.

You do NOT need to turn in your solutions for the following problems. However, I strongly encourage you to work through them.

1. Consider a multiple access wireless system with N mobile users, in which multiple users *simultaneously* communicate with a single base-station as illustrated in Fig. At the base-station, M antennas are equipped to achieve a more reliable receiving of the signals transmitted from N mobile users. The base-station tries to detect the symbols transmitted by each of the users using the received signals from the multiple antennas. Let h_{ij} be the channel gain from user i to the jth antenna. At the jth antenna, the received signal Y_j is corrupted by $\mathcal{N}(0, \sigma^2)$ additive Gaussian noise Z_j, independent across the antennas. The transmitted symbol X_i from user i is equally likely to be -1 or 1, independent among all users and independent of the noise. Furthermore, we assume the base-station has the knowledge of h_{ij} for all i, j.

   (a) An explicit mathematical model that describes the relation between the received signal Y_j and transmitted X_i is given by

   $$Y_j = h_{1j}X_1 + \sum_{i=2}^{N} h_{ij}X_i + Z_j, \quad j = 1, 2, \cdots, M$$

   Collectively, we have a vector representation

   $$\mathbf{y} = \underbrace{\mathbf{h}_1 X_1}_{\triangleq \mathbf{s}} + \underbrace{\sum_{i=2}^{N} \mathbf{h}_i X_i}_{\triangleq \mathbf{i}} + \mathbf{z},$$

   where $\mathbf{y} = [Y_1, Y_2, \cdots, Y_M]^T$, $\mathbf{h}_i = [h_{i1}, \cdots, h_{iM}]^T$ and $\mathbf{z} = [Z_1, \cdots, Z_M]^T$. Note that the model assumes the user 1's signal $\mathbf{s}$ is considered as the desired signal and those from the remaining users are considered as interference $\mathbf{i}$. That is we can write

   $$\mathbf{y} = \mathbf{s} + \mathbf{i} + \mathbf{z}.$$

   Determine the covariance matrix $\mathbf{K}$ of the interference-plus-noise vector $\mathbf{i} + \mathbf{z}$.

   (b) What is the optimal detector in terms of minimum error probability for X_1 if there were only 1 user in the system? Suppose for simplicity you decide to use this detector even in the multiuser environment. Give an expression for the average probability of error for user 1.

(c) This problem deals with performing detection using MMSE estimation. Consider a receiver which first obtains an LMMSE estimate of X_1 from the received signals at the antennas and then decides a 1 if the estimate is positive and -1 otherwise. Derive the decision rule for LMMSE receiver.

(d) The ***optimum combining*** filter $\mathbf{w}_{op}$ is defined to be the vector $\mathbf{w}$ that maximizes the signal to interference-plus-noise ratio (SINR) defined by

$$\text{SINR} \triangleq \frac{E\left[\left|\mathbf{w}^H\mathbf{s}\right|^2\right]}{E\left[\left|\mathbf{w}^H(\mathbf{i}+\mathbf{z})\right|^2\right]}.$$

Derive the expression of $\mathbf{w}_{op}$. And find the corresponding SINR. (Hint: Cauchy-Schwartz inequality)

(e) If the interference-plus-noise were considered as Gaussian random vector with zero mean and covariance matrix $\mathbf{K}$, find the probability of error in part (b) and part (c).