Homework 4

Solutions

Reading assignments:

1. Sec. 6.1 $\sim$ Sec. 6.7, textbook.
2. Sec. 7.1$\sim$ Sec. 7.6, textbook.

Problem assignments:

1. (a) Suppose that among the n games, the gambler wins k games and loses $n-k$, which gives $r = 2k - n$. Thus,

$$P\big[X[n] = rs\big] = \begin{cases} \binom{n}{\frac{n+r}{2}} p^{\frac{n+r}{2}} q^{\frac{n-r}{2}}, & \text{if } \frac{n+r}{2} \text{ an integer, } r \le n \\ 0, & \text{otherwise.} \end{cases}$$

(b) The mean function is

$$E\big[X[n]\big] = \sum_{i=1}^{n} E[W[i]] = ns(p-q).$$

The variance function is

$$\text{Var}\big(X[n]\big) = E\big[X^2[n]\big] - \big(E\big[X[n]\big]\big)^2,$$

for which we find $E[X^2[n]]$ first.
The second moment is

$$\begin{aligned} E\big[X^2[n]\big] &= E\left[\left(\sum_{i=1}^{n} W[i]\right)^2\right] \\ &= \sum_{i=1}^{n} E\left[W^2[i]\right] + \sum_{i \ne j} E\left[W[i]W[j]\right] \\ &= ns^2 + n(n-1)s^2(p-q)^2. \end{aligned}$$

Thus, the variance function is

$$\begin{aligned} \text{Var}\big(X[n]\big) &= E\big[X^2[n]\big] - \big(E\big[X[n]\big]\big)^2 \\ &= ns^2\left(1-(p-q)^2\right). \end{aligned}$$

Finally, the autocorrelation function for $m > n$ is

$$\begin{aligned} R_{XX}[m,n] &= E\big[X[m]X^*[n]\big] \\ &= E\Big[\big(X[m] - X[n] + X[n]\big)X[n]\Big] \\ &= E\big[X[m] - X[n]\big]E\big[X[n]\big] + E\big[X^2[n]\big] \quad \text{(indep. increment)} \\ &= (m-n)s(p-q) \cdot ns(p-q) + ns^2 + n(n-1)s^2(p-q)^2 \\ &= ns^2(1-(p-q)^2) + mns^2(p-q)^2. \end{aligned}$$

Therefore, we have the autocorrelation function

$$R_{XX}[m,n] = \min(m,n)s^2(1-(p-q)^2) + mns^2(p-q)^2.$$

(c) Let E be the event that the gambler will end up with N dollars with initial possession of K dollars. And let H denote the event that the gambler wins the first game. Then, using the total probability, the probability P_K is

$$\begin{aligned} P_K = P[E] &= P[E|H]P[H] + P[E|H^c]P[H^c] \\ &= pP[E|H] + qP[E|H^c]. \end{aligned}$$

By the independence of successive games, having won the first game is the same as if he were just starting but with $K+1$ dollars, so that $P[E|H] = P_{K+1}$. And, similarly, $P[E|H^c] = P_{K-1}$. Then, we have

$$P_K = pP_{K+1} + qP_{K-1}, \tag{1}$$

where the boundary conditions P_0 and P_N means the probability that the gambler ends up with N dollars if he starts with 0 and N dollars, respectively. It is clear to see that the boundary conditions satisfy $P_0 = 0$ and $P_N = 1$. So our objective now is to solve equation (1), which can be further rearranged to

$$P_{K+1} - P_K = \frac{q}{p}(P_K - P_{K-1}) \quad K = 1, 2, \ldots, N-1.$$

Solving for the above difference equation with the boundary conditions leads to

$$P_K = \begin{cases} \frac{1-(q/p)^K}{1-(q/p)^N} & \text{if } p \neq \frac{1}{2}, \\ \frac{K}{N} & \text{if } p = \frac{1}{2}. \end{cases}$$

(d) Following the assumptions in part (c). By symmetry and replacing p by q and K by $N-K$, the probability Q_K that the gambler will end up having no money left is

$$Q_K = \begin{cases} \frac{1-(p/q)^{N-K}}{1-(p/q)^N} & \text{if } q \neq \frac{1}{2}, \\ \frac{N-K}{N} & \text{if } q = \frac{1}{2}. \end{cases}$$

Please note that $P_K + Q_K = 1$, which means with probability one the gambler either loses all his money or accumulates N dollars. The probability that the games continue forever indefinitely with the gambler's fortune swing between 1 and $N-1$ is zero.

(e) This problem is the famous ***gambler ruin problem***. The gambler will very likely go bankrupt even by playing fair games ($p = q = 1/2$) if he greedily sets a very large N with only a small initial K. We can see this by letting N very very large and K finite, the probability Q_K will be very small even when $q = 1/2$. This in some sense tells us that we are unlikely to earn big fortune in a casino if we only have very little money (small K), ***unless*** we have a winning strategy in a way to make p larger that $1/2$.

2—7 See below.

Extra Problems

1. (a) The covariance matrix $\mathbf{K}$ of the interference-plus-noise vector $\mathbf{i}+\mathbf{z}$ is

$$E[(\mathbf{i}+\mathbf{z})(\mathbf{i}+\mathbf{z})^T] = \sum_{i=2}^{N} \mathbf{h}_i\mathbf{h}_i^T + \sigma^2\mathbf{I}.$$

(b) The optimal detector in terms of minimum error probability for X_1 if there were only 1 user in the system is given by the MAP criterion, and can be obtained as

$$\mathbf{h}_1^T\mathbf{y} \underset{\hat{X}_1=-1}{\overset{\hat{X}_1=1}{\gtrless}} 0.$$

The average probability of error P_e for user 1 is

$$\begin{aligned}
P_e &= P[\mathbf{h}_1^T\mathbf{y} < 0|X_1 = 1]P[X_1 = 1] + P[\mathbf{h}_1^T\mathbf{y} > 0|X_1 = -1]P[X_1 = -1] \\
&= P[\mathbf{h}_1^T\mathbf{y} < 0|X_1 = 1] \\
&= P\left[\mathbf{h}_1^T\mathbf{z} < -\left(||\mathbf{h}_1||^2 + \sum_{i\neq 1}\mathbf{h}_1^T\mathbf{h}_iX_i\right)\right] \\
&= E\left[P\left[\mathbf{h}_1^T\mathbf{z} < -\left(||\mathbf{h}_1||^2 + \sum_{i\neq 1}\mathbf{h}_1^T\mathbf{h}_iX_i\right)\middle|\{X_i \in \{-1,1\}, i\neq 1\}\right]\right] \\
&= \sum_{\{X_i\in\{-1,1\},i\neq 1\}} Q\left(\frac{||\mathbf{h}_1||^2 + \sum_{i\neq 1}\mathbf{h}_1^T\mathbf{h}_iX_i}{\sigma||\mathbf{h}_1||}\right)2^{1-N}.
\end{aligned}$$

(c) The LMMSE estimate of X_1 is

$$\hat{X}_{1_{lm}}(\mathbf{y}) = E[X_1] + K_{X_1\mathbf{y}}\mathbf{K}_\mathbf{y}^{-1}(\mathbf{y} - E[\mathbf{y}]),$$

where

$$\begin{aligned}
K_{X_1\mathbf{y}} &= E[X_1\mathbf{y}^T] = E[X_1(X_1\cdot\mathbf{h}_1 + \mathbf{i} + \mathbf{z})^T] = \mathbf{h}_1^T \\
\mathbf{K}_\mathbf{y} &= E[\mathbf{y}\mathbf{y}^T] \\
&= \sum_{i=1}^{N}\mathbf{h}_i\mathbf{h}_i^T + \sigma^2\mathbf{I},
\end{aligned}$$

where $\mathbf{I}$ is the $M\times M$ identity matrix. Note that $\mathbf{K}_\mathbf{y}$ is nonsingular since $\mathbf{x}^T\mathbf{K}_\mathbf{y}\mathbf{x} > 0$ for all nonzero vector $\mathbf{x}$. Thus,

$$\hat{X}_{1_{lm}}(\mathbf{y}) = \mathbf{h}_1^T\left(\sum_{i=1}^{N}\mathbf{h}_i\mathbf{h}_i^T + \sigma^2\mathbf{I}\right)^{-1}\mathbf{y}.$$

Let $\mathbf{u}^T = \mathbf{h}_1^T\left(\sum_{i=1}^{N}\mathbf{h}_i\mathbf{h}_i^T + \sigma^2\mathbf{I}\right)^{-1}$. Then the decision rule is

$$\mathbf{u}^T\mathbf{y} \underset{\hat{X}_1=-1}{\overset{\hat{X}_1=1}{\gtrless}} 0.$$

(d) The ***optimum combining*** filter $\mathbf{w}_{op}$ is the vector $\mathbf{w}$ that maximizes the signal to interference-plus-noise ratio (SINR) defined by

$$\begin{aligned} \text{SINR} &= \frac{E[|\mathbf{w}^H\mathbf{s}|^2}{E[|\mathbf{w}^H(\mathbf{i}+\mathbf{z})|^2} \\ &= \frac{\mathbf{w}^H\mathbf{h}_1\mathbf{h}_1^H\mathbf{w}}{\mathbf{w}^H\mathbf{K}\mathbf{w}}. \end{aligned}$$

As we did in HW#3, the maximum SINR is achieved when

$$\begin{aligned} \mathbf{w} &= \mathbf{K}^{-1}\mathbf{h}_1 = \left(\sum_{i=2}^{N}\mathbf{h}_i\mathbf{h}_i^T + \sigma^2\mathbf{I}\right)^{-1}\mathbf{h}_1 \\ &\triangleq \mathbf{w}_{op}. \end{aligned}$$

The maximum SINR achieved is $\mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1$.

(e) — For part (b), the error probability is

$$\begin{aligned} P[e] &= P[e|X_1=1] \\ &= P[\mathbf{h}_1^T\mathbf{y} < 0|X_1=1] \\ &= P\left[\mathbf{h}_1^T(\mathbf{i}+\mathbf{z}) < -||\mathbf{h}_1||^2\right], \end{aligned}$$

where $\mathbf{i}+\mathbf{z}$ is now considered as $N(\mathbf{0},\mathbf{K})$. By the definition of jointly Gaussian, it is clear that $\mathbf{h}_1^T(\mathbf{i}+\mathbf{z})$ is a Gaussian random variable with zero mean and variance $\mathbf{h}_1^T\mathbf{K}\mathbf{h}_1$. Thus, we have

$$\begin{aligned} P[e] &= P\left[\mathbf{h}_1^T(\mathbf{i}+\mathbf{z}) < -||\mathbf{h}_1||^2\right] \\ &= P\left[\frac{\mathbf{h}_1^T(\mathbf{i}+\mathbf{z})}{\sqrt{\mathbf{h}_1^T\mathbf{K}\mathbf{h}_1}} < -\frac{||\mathbf{h}_1||^2}{\sqrt{\mathbf{h}_1^T\mathbf{K}\mathbf{h}_1}}\right] \\ &= Q\left(\frac{||\mathbf{h}_1||^2}{\sqrt{\mathbf{h}_1^T\mathbf{K}\mathbf{h}_1}}\right). \end{aligned}$$

— For part (c), similarly, the error probability is

$$\begin{aligned} P[e] &= P[e|X_1=1] \\ &= P[\mathbf{u}^T\mathbf{y} < 0|X_1=1] \\ &= P\left[\mathbf{u}(\mathbf{i}+\mathbf{z}) < -\mathbf{u}^T\mathbf{h}_1\right], \end{aligned}$$

where $\mathbf{i}+\mathbf{z}$ is now considered as $N(\mathbf{0},\mathbf{K})$. By the definition of jointly Gaussian, it is clear that $\mathbf{u}^T(\mathbf{i}+\mathbf{z})$ is a Gaussian random variable with zero mean and variance $\mathbf{u}^T\mathbf{K}\mathbf{u}$. Thus, we have

$$P[e] = Q\left(\frac{\mathbf{h}_1^T\mathbf{K}_{\mathbf{y}}{}^{-1}\mathbf{h}_1}{\sqrt{\mathbf{u}^T\mathbf{K}\mathbf{u}}}\right). \tag{2}$$

We can further simplify (2) using matrix inversion lemma, which states for non-singular matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$, and any matrices $\mathbf{C}$, the following formula holds true

$$\begin{aligned} \mathbf{A} &= \mathbf{B} + \mathbf{C}\mathbf{D}\mathbf{C}^H \\ \mathbf{A}^{-1} &= \mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{C}\left(\mathbf{D}^{-1} + \mathbf{C}^H\mathbf{B}^{-1}\mathbf{C}\right)^{-1}\mathbf{C}^H\mathbf{B}. \end{aligned} \tag{3}$$

The covariance matrix of $\mathbf{K_y}$ can be represented by

$$\begin{aligned}\mathbf{K_y} &= \sum_{i=1}^{N} \mathbf{h}_i \mathbf{h}_i^T + \sigma^2 \mathbf{I} \\ &= \mathbf{K} + \mathbf{h}_1 \mathbf{h}_1^T.\end{aligned}$$

Letting $\mathbf{B} = \mathbf{K},\ \mathbf{C} = \mathbf{h}$ and $\mathbf{D} = 1$, and substituting into (3), we have

$$\mathbf{K_y}^{-1} = \mathbf{K}^{-1} - \frac{\mathbf{K}^{-1}\mathbf{h}_1\mathbf{h}_1^T\mathbf{K}^{-1}}{1 + \mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}.$$

Note that the denominator of the 2nd term in the RHS is just a scalar. With this, the LMMSE weighting vector $\mathbf{u}$ can be rewritten as

$$\begin{aligned}\mathbf{u} &= \mathbf{K_y}^{-1}\mathbf{h}_1 \\ &= \left(\mathbf{K}^{-1} - \frac{\mathbf{K}^{-1}\mathbf{h}_1\mathbf{h}_1^T\mathbf{K}^{-1}}{1 + \mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}\right)\mathbf{h}_1 \\ &= \mathbf{K}^{-1}\left(\mathbf{h}_1 - \frac{\mathbf{h}_1\mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}{1 + \mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}\right) \\ &= \frac{\mathbf{K}^{-1}\mathbf{h}_1}{1 + \mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}.\end{aligned}$$

With further algebraic efforts, we can find

$$\begin{aligned}\mathbf{h}_1^T\mathbf{K_y}^{-1}\mathbf{h}_1 &= \frac{\mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}{1 + \mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1} \\ \mathbf{u}^T\mathbf{K}\mathbf{u} &= \frac{\mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}{(1 + \mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1)^2}.\end{aligned}$$

Hence, we have the error probability for the LMMSE receiver

$$P[e] = Q\left(\sqrt{\mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1}\right),$$

where $\mathbf{h}_1^T\mathbf{K}^{-1}\mathbf{h}_1$ is the maximum SINR achieved in part (d). This implies that the LMMSE receiver and Optimum Combining receiver are actually equivalent in terms of BER when interference-plus-noise is considered as Gaussian, which an important result in practical applications.

6.23

a) $\mu_X[n] = E[A\cos\omega n + B\sin\omega n]$
$= E[A]\cdot\cos\omega n + E[B]\cdot\sin\omega n$
$= 0 \cdot \text{''} + 0\cdot\text{''}$
$= 0$, a constant.

$$E[X[n+m]X[n]] =$$
$$= E[(A\cos\omega(n+m) + B\sin\omega(n+m))(A\cos\omega n + B\sin\omega n)]$$
$$= \sigma^2(\cos\omega(n+m)\cos\omega n + \sin\omega(n+m)\sin\omega n)$$
$$= \sigma^2\cos(\omega(n+m) - \omega n)$$
$$= \sigma^2\cos\omega m = R_X[m]. \quad \therefore \text{ WSS}.$$

b) Consider $E[X^3[n]] = E[(A\cos\omega n + B\sin\omega n)^3]$
$= E[A^3]\cdot\cos^3\omega n + E[A^2B](\cdot)$
$+ E[AB^2](\cdot) + E[B^3]\cdot\sin^3\omega n$

Now $E[A^2B] = E[A^2]\cdot E[B] = \sigma^2\cdot 0 = 0 = E[AB^2]$

so
$$E[X^3[n]] = \underset{\substack{\uparrow\\ \text{Third-order}\\ \text{moment } (\neq 0)}}{\xi_3}\cdot\underbrace{(\cos^3\omega n + \sin^3\omega n)}_{\neq\ \text{constant}}$$

Hence $X[n]$ cannot be stationary in the strict sense.

(6.24) a) $R_X(0)$ must be non negative & $K_{\underline{X}}[0] = P^2 - \mu^2$

so $P^2 \geq \mu^2$. Set $\sigma^2 \triangleq P^2 - \mu^2 \ (\geq 0)$.

b) $$\underline{\underline{K}} = \begin{bmatrix} \sigma^2 & & -\mu^2 \\ & \ddots & \\ -\mu^2 & & \sigma^2 \end{bmatrix}$$ so $\underline{a}\,\underline{\underline{K}}\,\underline{a}^{\dagger} = N\sigma^2 - N(N-1)\mu^2 \geq 0$

(since we take $\underline{a} = \underline{1}$.)

so $\mu^2 \leq \dfrac{N\sigma^2}{N(N-1)} = \dfrac{\sigma^2}{N-1}$.

c) By taking a sequence of ever increasing N values, we conclude μ must be zero, for finite P^2 and hence σ^2.

6.36 (a) Since $\{0\} = [0,1] - (0,1]$

hence $P[\{0\}] = P[0,1] - P(0,1]$

or $P[\{0\}] = P[\Omega] - P(0,1]$

$P[\{0\}] = 1 - 1 = 0$

(b)

i) $X[n,\zeta] = e^{-n\zeta}$

$$E\left[|X[n,\zeta] - 0|^2\right] = E\left[e^{-2n\zeta}\right]$$

$$= \int_0^1 2 \cdot e^{-2n\zeta} d\zeta = \frac{e^{-2n} - 1}{-2n}$$

Let $n \to \infty$, then $\frac{e^{-2n} - 1}{-2n} \longrightarrow 0$.

therefore $X[n,\zeta]$ converges in mean-square sense.

Also $\lim_{n\to\infty} X[n,\zeta] = \begin{cases} 1, & \zeta = 0 \\ 0, & 0 < \zeta \le 1 \end{cases}$ therefore $X[n]$ converges almost everywhere (a.e. or almost surely)

ii) $X[n,\zeta] = \sin(\zeta + \frac{1}{n}) \xrightarrow{?} \sin(\zeta)$

$$E\left\{|\sin(\zeta + \tfrac{1}{n}) - \sin\zeta|^2\right\}$$

$$= E\left\{|\sin\zeta \cos\tfrac{1}{n} + \cos\zeta \sin\tfrac{1}{n} - \sin\zeta|^2\right\}$$

$$= E\left\{|\sin\zeta[\cos\tfrac{1}{n} - 1] + \cos\zeta \sin\tfrac{1}{n}|^2\right\}$$

$$= E\left\{\sin^2\zeta[\cos^2\tfrac{1}{n} + 1 - 2\cos\tfrac{1}{n}] + \cos^2\zeta \sin^2\tfrac{1}{n} + 2\cos\zeta \sin\tfrac{1}{n} \sin\zeta[\cos\tfrac{1}{n} - 1]\right\}$$

$$= E\left\{\sin^2\zeta \cos^2\tfrac{1}{n} + \sin^2\zeta - 2\sin^2\zeta \cos\tfrac{1}{n} + \cos^2\zeta \sin^2\tfrac{1}{n} + 2\cos\zeta \sin\zeta \cos\tfrac{1}{n} \sin\tfrac{1}{n} - 2\cos\zeta \sin\zeta \sin\tfrac{1}{n}\right\}$$

If the expectation is performed (The integral is calculated), we see that: as $n \longrightarrow \infty$, the expected value goes to zero. Further $\sin(\zeta + \frac{1}{n}) \rightarrow \sin\zeta$, $\forall \zeta \in S$

Therefore, the convergence is everywhere also.

iii) $X[n,\zeta] = \cos^n \zeta, \quad \zeta \in [0,1]$

$$\cos\zeta = \begin{cases} 1, & \zeta = 0 \\ <1, & 0<\zeta\le 1 \end{cases}, \quad \text{Therefore}$$

$$\lim_{n\to\infty} X[n,\zeta] = \lim_{n\to\infty} \cos^n\zeta = \begin{cases} 1, & \zeta=0 \\ 0, & 0\le\zeta\le 1 \end{cases}$$

Hence, it converges almost everywhere.

$E\{|\cos^n(\zeta) - 0|^2\} = E[\cos^{2n}\zeta]$

however, as $n\to\infty$, $E[\cos^{2n}\zeta] \longrightarrow 0$

hence, it converges in m-s. sense also.

(c) i) $X[n,\zeta] = e^{-n\zeta} \xrightarrow[n\to\infty]{} 0 \quad$ (a.s.)

ii) $X[n,\zeta] = \sin(\zeta + \frac{1}{n}) \xrightarrow[n\to\infty]{} \sin\zeta$

iii) $X[n,\zeta] = \cos^n\zeta \xrightarrow[n\to\infty]{} 0 \quad$ (a.s.)

6.37 (i)

If $X[n] \longrightarrow X$ in mean-square, then X is independent of $X[n]$ since $X[n]$ is a sequence of jointly independent random variables.

$E\{|X[n]-X|^2\} = E|X^2[n]| - 2E[X[n]X] + E\{X^2\}$

$\because X$ independent of $X[n]$

$\Rightarrow E\{|X[n]-X|^2\} = E\{X^2[n]\} - 2E\{X[n]\}E[X] + E[X^2]$

Since X is a gaussian random variable with mean σ and variance σ^2.

$\because \lim_{n\to\infty} E\{X^2[n]\} = \sigma^2 + \sigma^2 = 2\sigma^2$

$\lim_{n\to\infty} E\{X[n]\} = \sigma$

$\therefore \quad E[X] = \sigma$

$$E[X^2] = 2\sigma^2$$

$$\Rightarrow \lim_{n\to\infty} E[|X[n]-X|^2] = \lim_{n\to\infty} E[X^2[n]] - 2\lim_{n\to\infty} E[X[n]]\cdot E[X] + E[X^2]$$

$$\Rightarrow \lim_{n\to\infty} E[|X[n]-X|^2] = 2\sigma^2 - 2\sigma\cdot\sigma + 2\sigma^2 = 2\sigma^2 \neq 0.$$

So, it doesn't converge in mean-square sense.

(ii) $X[n]$ does not converge in probability.

(iii) $X[n]$ clearly converges in pdf and PDF or distribution with pdf given as,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{1}{2\sigma^2}(x-\sigma)^2, \quad -\infty < x < +\infty.$$

7.8 (a) We know

$$f_T(t;n) = \underbrace{f_T(t) * \cdots * f_T(t)}_{n \text{ times}}$$

with $\quad f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0$

therefore

$$f_T(t;2) = f_T(t) * f_T(t) = \int_0^t (\lambda e^{-\lambda u}) \cdot (\lambda e^{-\lambda(t-u)})\, du$$

$$= \int_0^t \lambda^2 e^{-\lambda t}\, du = \lambda^2 t e^{-\lambda t} u(t),$$

and $\quad f_T(t;3) = f_T(t;2) * \lambda e^{-\lambda t}$

$$= \int_0^t (\lambda^2 u e^{-\lambda u})(\lambda e^{-\lambda(t-u)})\, du$$

$$= \lambda^3 e^{-\lambda t} \int_0^t u\, du = \lambda^3 \frac{t^2}{2} e^{-\lambda t} u(t).$$

Hence for $f_T(t;n)$, we will get;

$$f_T(t;n) = \frac{\lambda^n t^{n-1}}{(n-1)!} \cdot e^{-\lambda t} u(t).$$

(b) $F_{\tau'_{[i]}}(\tau' | T=t) = P[\tau_{[i]} \leq \tau' + t \,|\, \tau_{[i]} \geq t]$

$$= \frac{\int_t^{t+\tau'} \lambda e^{-\lambda \tau} d\tau}{\int_t^{\infty} \lambda e^{-\lambda \tau} d\tau} = \frac{[-e^{-\lambda \tau}]_t^{t+\tau'}}{[-e^{-\lambda \tau}]_t^{\infty}}$$

$$= 1 - e^{-\lambda \tau'}$$

or: $f_{\tau'_{[i]}}(\tau' | T=t) = \lambda e^{-\lambda \tau'}$, same density as $\tau_{[i]}$.

(c) $$f_{\tau'_{[i]}}(\tau') = \int_{-\infty}^{+\infty} f_{\tau'_{[i]}}(\tau' | T=t)\, f_T(t)\, dt$$

but from part (b), the conditional pdf is independent of the variable t,

Thus,

$$f_{\tau'_{[i]}}(\tau') = f_{\tau_{[i]}}(\tau' | T=t) \int_{-\infty}^{+\infty} f_T(t)\, dt$$

$$= f_{\tau_{[i]}}(\tau') \qquad \text{using result of (b).}$$

7.17

(a) $X(t)=W^2(t)$, $f_W(w,t)=\frac{1}{\sqrt{2\pi t}}\exp^{(-\frac{w}{2t})^2}$, $t>0$

So, transformation is $x=w^2$,

with two roots at $w=+\sqrt{x}$ and $-\sqrt{x}$

So, $f_X(x;t)=f_W(+\sqrt{x})\cdot\frac{1}{|J|}+f_W(-\sqrt{x})\cdot\frac{1}{|J|}$

with $|J|=\left|\frac{dx}{dw}\right|=|2w|=2\sqrt{x}$

Thus, we get

$$f(x;t)=\frac{1}{\sqrt{2\pi t}}\cdot\exp^{(-\frac{x}{2t})}\cdot\frac{1}{2\sqrt{x}}\cdot 2$$

$$=\frac{1}{\sqrt{2\pi x t}}\cdot\exp^{(-\frac{x}{2t})}\quad t>0,\ x>0$$

$$=0\quad,\quad x\le 0$$

(b) We first find the joint density $f_X(x_2,x_1)$

$$f_X(x_2,x_1)=f_W(\sqrt{x_2},\sqrt{x_1})\cdot\frac{1}{|J|}+f_W(-\sqrt{x_2},\sqrt{x_1})\cdot\frac{1}{|J|}+f_W(\sqrt{x_2},-\sqrt{x_1})\frac{1}{|J|}+f_W(-\sqrt{x_2},-\sqrt{x_1})\cdot\frac{1}{|J|}$$

$$|J|=\begin{vmatrix}2w, & 0\\ 0 & 2w_2\end{vmatrix}=4\sqrt{x_1x_2}$$

So, $f_X(x_2,x_1)=f_{W_1}(\sqrt{x_1})f_{W_2-W_1}(\sqrt{x_2}-\sqrt{x_1})\cdot\frac{1}{4\sqrt{x_2x_1}}$

$+f_{W_1}(\sqrt{x_1})f_{W_2-W_1}(-\sqrt{x_2}-\sqrt{x_1})\cdot\frac{1}{4\sqrt{x_2x_1}}$

$+f_{W_1}(-\sqrt{x_1})f_{W_2-W_1}(\sqrt{x_2}+\sqrt{x_1})\cdot\frac{1}{4\sqrt{x_2x_1}}$

$+f_{W_1}(-\sqrt{x_1})f_{W_2-W_1}(-\sqrt{x_2}+\sqrt{x_1})\cdot\frac{1}{4\sqrt{x_2x_1}}$

$$=\frac{1}{2\pi}\frac{1}{\sqrt{x_1t_1x_2(t_2-t_1)}}\left[\frac{1}{2}e^{(-\frac{x_1}{2t_1}-\frac{(\sqrt{x_2}-\sqrt{x_1})^2}{2(t_2-t_1)})}+\frac{1}{2}e^{(-\frac{x_1}{2t_1}-\frac{(\sqrt{x_2}+\sqrt{x_1})^2}{2(t_2-t_1)})}\right]$$

$t_2>t_1$, $x_2,x_1>0$

Then,

$$f(x_2|x_1) = f(x_2,x_1)/f(x_1)$$

$$= \frac{1}{\sqrt{2\pi x_2(t_2-t_1)}}\left[\frac{1}{2}e^{-\frac{(\sqrt{x_2}-\sqrt{x_1})^2}{2(t_2-t_1)}}+\frac{1}{2}e^{-\frac{(\sqrt{x_2}+\sqrt{x_1})^2}{2(t_2-t_1)}}\right]$$

$$t_2 > t_1,\ x_2, x_1 > 0$$

(c) Yes, it is Markov.

Consider $f(x_3,x_2,x_1;t_3,t_2,t_1) = f(x_3,x_2,x_1)$

Calculation shows that

$$f(x_3,x_2,x_1) = \frac{1}{8\sqrt{x_3x_2x_1}}f_W(\sqrt{x_1})\cdot\left[f_{\Delta W}(\sqrt{x_2}-\sqrt{x_1})+f_{\Delta W}(\sqrt{x_2}+\sqrt{x_1})\right]$$

$$\cdot\left[f_{\Delta W}(\sqrt{x_3}-\sqrt{x_2})+f_{\Delta W}(\sqrt{x_3}+\sqrt{x_2})\right]$$

and from part (b)

$$f(x_2,x_1) = \frac{1}{4\sqrt{x_2x_1}}f_W(\sqrt{x_1})\left[f_{\Delta W}(\sqrt{x_2}-\sqrt{x_1})+f_{\Delta W}(\sqrt{x_2}+\sqrt{x_1})\right]$$

So,

$$f(x_3|x_2x_1) = \frac{1}{2\sqrt{x_3}}\left[f_{\Delta W}(\sqrt{x_3}-\sqrt{x_2})+f_{\Delta W}(\sqrt{x_3}+\sqrt{x_2})\right]$$

$$= f(x_3|x_2) \text{ from part (b),}$$

which is consistent with the definition of a Markov Process.

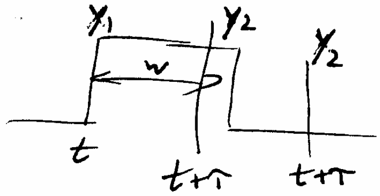
(d) From part (a) and part (b), we know

$$f_X(x_1,x_2;t_2,t_1) \neq f_X(x_2-x_1;t_2,t_1)\cdot f_X(x_1;t_1)$$

So, $X(t)$ doesn't have independent increments.

7.34 $R_Y(\tau) = E[Y(t)\,Y(t+\tau)] = E_W\big[E[Y(t)Y(t+\tau)\,|\,W]\big]$

Let $W=w$ & evaluate $E[Y(t)\,Y(t+\tau)\,|\,W=w]$. Since not a function of t, can take t at start of pulse (by WSS of Y)



Now for $\tau < w$ $\quad Y_1 = Y_2$

so $Y_1 Y_2 = Y_1^2$

for $\tau > w$ $\quad Y_1 Y_2$ with $Y_1 \perp Y_2$

$$E[Y(t)\,Y(t+\tau)\,|\,W=w] = E[Y_1Y_2] = E[Y_1]E[Y_2] = E^2[Y], \quad \tau > w$$
$$= E[Y_1^2] = E[Y^2], \quad \tau < w$$

or $\quad E[Y(t)\,Y(t+\tau)\,|\,W=w] = \begin{cases} E^2[Y] & \tau > w \\ E[Y^2] & \tau < w \end{cases}$

so

$$E\big[E[Y(t)Y(t+\tau)\,|\,W]\big] = E^2[Y]\,P[W<\tau] + E[Y^2]\,P[W>\tau]$$

& $E[X]=0$, $P[W>\tau] = \int_\tau^\infty \lambda e^{-\lambda w}\,dw = e^{-\lambda\tau}$

$$E[Y^2] = \int_{-\infty}^{+\infty} y^2 f_Y(y)\,dy = \sigma^2 \quad (E[Y]=0 \text{ since the process is zero mean}$$

$$\therefore R_Y(\tau) = \sigma^2 e^{-\lambda\tau}, \quad \tau > 0$$

Since $R_Y(\tau)$ must be even in τ, we have

$$R_Y(\tau) = \sigma^2 \exp -\lambda|\tau| \quad \text{for } -\infty < \tau < +\infty.$$

$$S_Y(\omega) = \frac{2\lambda\sigma^2}{\omega^2 + \lambda^2} \quad \text{for } -\infty < \omega < +\infty$$

$\tau = 0$, $R_Y(0) = \sigma^2$ = mean square value of X, as it should be.

$\tau = \infty$, $R_Y(\infty) = 0$ = mean value of the process squared. Separated elements become uncorrelated.

7.38 (a) $\mu_X(t) = E[N\cos(2\pi f_0 t + \Theta)]$

$= E[N]\, E[\cancel{\cos(2\pi f_0 t + \Theta)}]$ 0

$\Longrightarrow \mu_X(t) = 0$

(b) $K_X(t,s) = R_X(t,s)$, since $\mu_X(t) = 0,\ \forall t$

$= E[N^2 \cos(2\pi f_0 t + \Theta)\cos(2\pi f_0 s + \Theta)]$

$= E[N^2] \int_{-\pi}^{\pi} \cos(2\pi f_0 t + \theta)\cos(2\pi f_0 s + \theta)\frac{d\theta}{2\pi}$

$= E[N^2] \int_{-\pi}^{+\pi} \cos(2\pi f_0 (t-s) + \theta)\cos\theta\, \frac{d\theta}{2\pi}$

We remember:

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

So, $\cos(2\pi f_0 (t-s) + \theta)\cos\theta$

$= \frac{1}{2}\cos(2\pi f_0 (t-s) + 2\theta) + \frac{1}{2}\cos 2\pi f_0 (t-s)$

Now, $\int_{-\pi}^{+\pi} \cos 2\theta \frac{d\theta}{2\pi} = 0$ & $\int_{-\pi}^{+\pi} 1 \cdot \frac{d\theta}{2\pi} = 1$

So, $\int_{-\pi}^{+\pi} \cos(2\pi f_0 (t-s) + \theta)\cos\theta \frac{d\theta}{2\pi} = \frac{1}{2}\cos 2\pi f_0 (t-s)$

For the poisson r.v.; we have $E[N^2] = \lambda + \lambda^2$ when $\lambda = Var(N)$ & $\lambda^2 = (\mu_N)^2$. Thus

$K_X(t,s) = \frac{1}{2}(\lambda + \lambda^2)\cos 2\pi f_0 (t-s)$

(c) By definition of w.s.s., the answers to the (a) and (b) show that $X(t)$ is w.s.s.

(d) To show strict sense stationarity, we must show that for any m, the mth order distribution function: