Stochastic Processes

Midterm 1

9:00 a.m. – 11:30 a.m., 10/27/08

IMPORTANT:

- Remember to write down your i.d. number and your name.
- There are 4 problem sets with 110 points in total.
- Show all your work with detailed explanations. Correct answers ***without*** any explanations will carry **NO** credits. However, wrong answers with correct reasonings will get partial credits.

You may need the following formulas:

- Let $\mathbf{x} = [X_1, X_2, \cdots, X_n]$ be a real Gaussian random vector with mean vector $\mathbf{m}_x$ and covariance matrix $\mathbf{K}_x$. Then, the joint pdf $f_{\mathbf{x}}(\boldsymbol{x})$ is given by

$$f_{\mathbf{x}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}\det(\mathbf{K}_x)^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \mathbf{m}_x)^T \mathbf{K}_x^{-1}(\boldsymbol{x} - \mathbf{m}_x)\right).$$

- Consider a jointly Gaussian random vector $[\mathbf{x}^T\ \mathbf{y}^T]^T$ with $\mathbf{x} \sim \mathcal{N}(\mathbf{m}_\mathbf{x}, \mathbf{K}_\mathbf{x})$ and $\mathbf{y} \sim \mathcal{N}(\mathbf{m}_\mathbf{y}, \mathbf{K}_\mathbf{y})$. Then, the conditional density $f_{\mathbf{x}|\mathbf{y}}(\mathbf{x}|\mathbf{y})$ also follows a joint Gaussian density with

$$\mathbf{x}|\mathbf{y} \sim \mathcal{N}\Big(\mathbf{m}_\mathbf{x} + \mathbf{K}_{\mathbf{xy}}\mathbf{K}_\mathbf{y}{}^{-1}(\mathbf{y} - \mathbf{m}_\mathbf{y}), \mathbf{K}_\mathbf{x} - \mathbf{K}_{\mathbf{xy}}\mathbf{K}_\mathbf{y}{}^{-1}\mathbf{K}_{\mathbf{yx}}\Big),$$

where $\mathbf{K}_{\mathbf{xy}} = E\left[(\mathbf{x} - \mathbf{m}_\mathbf{x})(\mathbf{y} - \mathbf{m}_\mathbf{y})^H\right]$ and $\mathbf{K}_{\mathbf{yx}} = \mathbf{K}_{\mathbf{xy}}^H$

- The correlation coefficient ρ between random variables X and Y is defined by

$$\rho \triangleq \frac{E\big[(X - \mu_X)(Y - \mu_Y)\big]}{\sigma_X \sigma_Y},$$

where $\mu_X = E[X]$, $\mu_Y = E[Y]$, $\sigma_X^2 = \mathrm{Var}(X)$, and $\sigma_Y^2 = \mathrm{Var}(Y)$.

1. (15 points)

   Jointly Gaussian implies marginal Gaussian, but the converse is not necessarily true. Please explain this statement, and justify it.

2. (10+10=20 points)

   Let X and Y be i.i.d. standard Gaussian random variables.

   (a) Show that $X + Y$ and $X - Y$ are independent.

   (b) Find $E[X^3 - Y^3 | X - Y]$. (Hint: Use part (a).)

3. (10 points $\times$ 5 = 50 points)

Let $\mathbf{w} = [X, Y, Z]^T \sim \mathcal{N}(\mathbf{m}_\mathsf{w}, \mathbf{K}_\mathsf{w})$, where $\mathbf{m}_\mathsf{w} = [1\ 2\ 0]^T$ and the covariance matrix $\mathbf{K}_\mathsf{w}$ is

$$\mathbf{K}_\mathsf{w} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) Find α such that $X + \alpha Z$ and Z are independent.

(b) Find the probability density function (pdf) for $S = X + Y + Z$.

(c) Find the conditional variance $\mathrm{Var}(X|Y, Z)$.

(d) Find the correlation coefficient ρ_{xy} between X and Y.

(e) Let $\mathbf{r} = [X, Z]^T$. Find $\mathbf{A}$ and $\mathbf{b}$ such that the linear transformation $\mathbf{A}\mathbf{r} + \mathbf{b}$ is a Gaussian random vector with uncorrelated components, each with zero mean and unit variance.

4. (10+15=25 points)

Consider a communication system that is corrupted by unknown interferece Z and Gaussian noise W. Mathematically, the received signal Y can be modeled as

$$Y = X + Z + W,$$

where X is the desired signal equally likely to 1 or -1, and $W \sim \mathcal{N}(0, \sigma^2)$. We are given that $E[Z] = 0$ and $\mathrm{Var}(Z) = A^2$, but the exact statistical distribution of Z is not known. Assume $X,\ Z,\ W$ are mutually independent.

(a) Suppose we model Z as a Gaussian random variable. Find the posterior probability $P[X = 1|Y = y]$.

(b) Suppose now Z is modeled as a binary discrete random variable equally likely to be A or $-A$. Again, find the posterior probability $P[X = 1|Y = y]$.