Stochastic Processes

Topic 7

Random Process

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Summary

In this topic, I will discuss:

- Definition of Random Processes
- Poisson Counting Process
- Random Processes in LTI Systems
- Power Spectral Density
- Markov Processes

Notation

We will use the following notation rules, unless otherwise noted, to represent symbols during this course.

- Boldface upper case letter to represent **MATRIX**
- Boldface lower case letter to represent **vector**
- Superscript $(\cdot)^T$ and $(\cdot)^H$ to denote transpose and hermitian (conjugate transpose), respectively
- Upper case italic letter to represent *RANDOM VARIABLE*

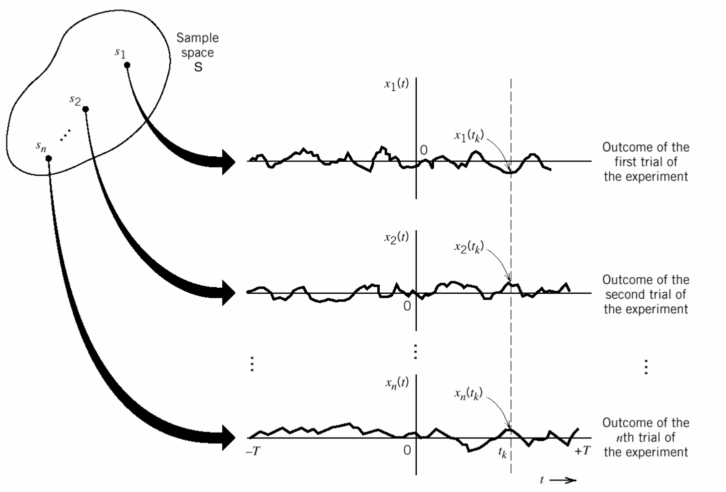

Figure 1: Illustration of a random process.

1 Random Process

(1) Definition (Random Process)

Let $\varepsilon \in \Omega$ be an outcome of the sample space Ω. Let $X(t, \varepsilon)$ be a mapping of the sample space Ω into a space of ***continuous time functions***. This mapping is called a random process if at each fixed time the mapping is a random variable.

Example:

Consider

$$X(t, \varepsilon) = A \cdot \cos(\omega_0 t + \theta(\varepsilon))$$

where $\theta(\varepsilon)$ is a random variable. Then, $X(t, \varepsilon)$ is a random process.

A random process can be simply regarded as *a function of time* with one or more *random parameters* in it.

Remarks:

— A ***random process*** is a 2-variable ***function*** that evolves in continuous time.

A ***random sequence*** is a 2-variable ***function*** that evolves in discrete time.

— For a fixed outcome ε, say ε_1, $X(t, \varepsilon_1)$ is called a ***sample function*** and is a non-random (***deterministic***) function. That is, once we know what the outcome ε is, the sample function associated with that ε is also determined.

— For fixed t, say t_1, $X(t_1)$ is a random variable.

— If we sample the random process at N times $t_1, \cdots, t_N$, we form a random vector $\Big[X(t_1), X(t_2), \cdots, X(t_N)\Big]^T$.

(2) A random process $X(t)$ is statistically specified by its complete set of nth order probability distribution (or density) function

$$F_X\Big(\ x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n\ \Big)$$

for all $x_1, x_2, \cdots, x_n$, and for all time $t_1 < t_2 < \cdots < t_n$.

(3) The mean function, autocorrelation function, and autocovariance function are defined as:
Mean function:

$$\mu_X(t) \triangleq E\Big[X(t)\Big] = \int_{-\infty}^{\infty} x f_X(x;t)dx,$$

Autocorrelation function: for all t_1 and t_2

$$\begin{aligned} R_X(t_1,t_2) &\triangleq E\Big[X(t_1)X^*(t_2)\Big] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^* f_X(x_2,x_2;t_1,t_2)dx_1dx_2, \end{aligned}$$

Autocovariance function: for all t_1 and t_2

$$\begin{aligned} K_X[t_1,t_2] &\triangleq E\Big[\big(X(t_1)-\mu_X(t_1)\big)\big(X(t_2)-\mu_X(t_2)\big)^*\Big] \\ &= R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2). \end{aligned}$$

(4) **Definition (Independent Increments)**

A random process is said to have ***independent increments*** if the set of n random variables

$$X(t_1), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$$

are jointly independent for $t_1 < t_2 < \cdots < t_n$ and for all $n > 1$.

Example: (**Random Sinusoid Waveform**) (p. 405)

Consider the random process

$$X(t) = A \cdot \sin(\omega_0 t + \Theta),$$

where A and Θ are independent random variables and Θ is uniformly distributed over $[-\pi, +\pi]$.

Example: (**Asynchronous Binary Signaling**)

We can model, in the absence of noise and interference, the continuous time received binary signal $X(t)$ of a communication link by the asynchronous binary signaling (ABS) process

$$X(t) = \sum_n X_n \cdot w\left(\frac{t - D - nT}{T}\right),$$

where $X_n \in [-1, +1]$ equally likely, D is the unknown delay typically modeled as a uniform random variable in $[-T/2, T/2)$, $w(t) = u(t + 1/2) - u(t - 1/2)$ is the signal pulse shaping waveform, and T is the symbol duration.

(Exercise) (p. 407)
What are the mean and autocorrelation function of $X(t)$?

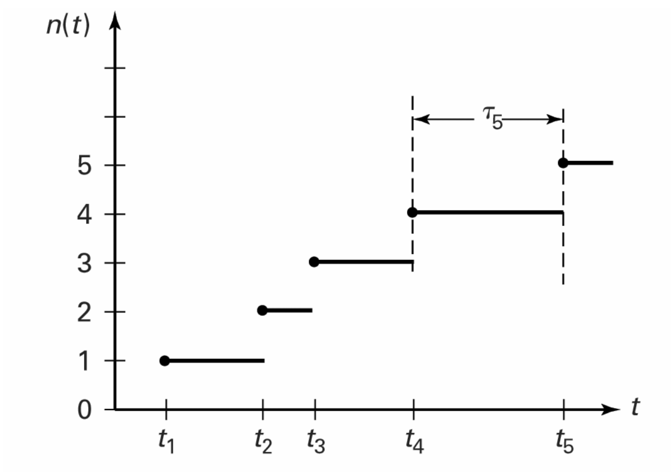

Figure 2: A sample function of the Poisson counting process.

2 Poisson Counting Process

(1) ***Poisson counting process*** can be used to model

— ***Number*** of customers arriving at a bank during a time interval (Management)

— ***Number*** of buses passing by a cross road (Traffic Control)

— ***Number*** of packets arriving at a buffer (Networking)

(2) In general, we can model the total ***number of "rare events"*** that have occurred up to time t by

$$N(t) = \sum_{n=1}^{\infty} u\big(t - T[n]\big),$$

where $u(t)$ is the unit step function and $T[n]$ is the waiting time sequence, i.e.

$$T[n] = \sum_{i=1}^{n} \tau[i]$$

is the ***waiting time*** to the nth occurrence with $\tau[i]$ being exponentially distributed with rate λ, which we have seen previously.

The probability mass function of the Poisson counting process $N(t)$ can be calculated by relating $N(t)$ with $T[n]$. To be more specific,

$$P\Big[N(t) = n\Big] = P\Big[T[n] \leq t,\ T[n+1] > t\Big].$$

The above probability mass function can be carried out as

$$\begin{aligned}
P\Big[N(t)=n\Big] &= P\Big[T[n]\le t,\ T[n+1]>t\Big] \\
&= \int_0^{\infty} P\Big[T[n]\le t,\ T[n]+\tau[n+1]>t\Big|T[n]=\alpha\Big]\cdot f_T(\alpha;n)d\alpha \\
&= \int_0^{t} P\Big[\tau[n+1]>t-\alpha\Big]\cdot f_T(\alpha;n)d\alpha \\
&= \int_0^{t} \frac{\lambda e^{-\lambda\alpha}(\lambda\alpha)^{n-1}}{(n-1)!}\left(\int_{t-\alpha}^{\infty}\lambda e^{-\lambda\beta}d\beta\right)d\alpha \\
&= \frac{(\lambda t)^n}{n!}e^{-\lambda t}u(t) \quad \text{for } t\ge 0,\quad n\ge 0,
\end{aligned}$$

which is the pmf of a Poisson random variable with mean λt.

Remarks:

— Poisson counting process has independent increments.

— For a time interval $(t_a,t_b]$, the increment $N(t_b)-N(t_a)$ has a pmf

$$P[N(t_b)-N(t_a)=n]=\frac{(\lambda(t_b-t_a))^n}{n!}e^{-\lambda(t_b-t_a)}.$$

— What is the autocorrelation function of $N(t)$? (p. 411)

Next, we will show the first 2 remarks:

Problem Formulation:
Let $N(t)$ be a Poisson counting process with rate λ.

(1) For any $s>0$, show that the increment $N(t+s)-N(s)$, $t>0$, is independent of $N(u)$ with all $u\le s$.

(2) Show that the increment $N(t+s)-N(s)$ in part (1) is also a Poisson process with rate λ.

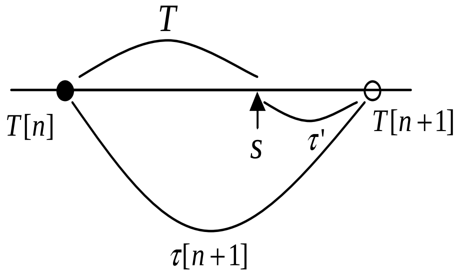

Figure 3: The relation between τ' and T.

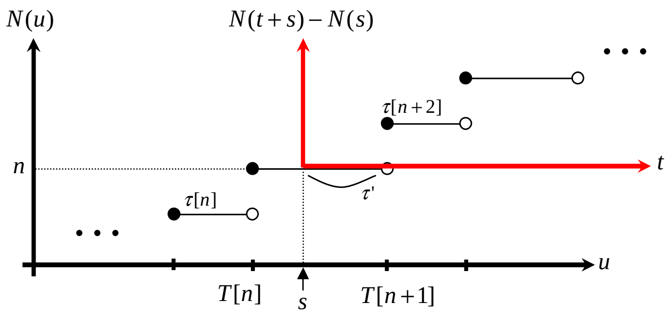

Figure 4: The illustration of the Poisson Counting Process $N(t+s) - N(s)$ for $s \geq 0$.

Proof:

(1) Without loss of generality, we assume $T[n] \leq s < T[n+1]$.

Let $T \triangleq s - T[n]$. Knowing $\{N(u) : u \leq s\}$ is equivalent to knowing the event

$$A \triangleq \{\tau[1], \tau[2] \ldots \tau[n], T \triangleq s - T[n]\}.$$

That is we know at what time points the Poisson process $N(u)$ has a jump for $u \leq s$. Similarly, knowing $\{N(t+s) - N(s) : s \geq 0\}$ is equivalent to knowing the event

$$B \triangleq \{\tau' \triangleq T[n+1] - s, \tau[n+1], \tau[n+2] \ldots\}.$$

So, if we can show that T and τ' are independent, then we know event A and event B are statistically independent (based on knowing that $\tau[n]$ are independent random sequence). Fig. 1 and Fig. 2 might be helpful in understanding the above statement.

From Problem 7.8 of the textbook, we see that τ' is indeed independent of T and is exponentially distributed with parameter λ. Therefore, we can say $\{N(u) : u \leq s\}$ and $\{N(t+s) - N(s) : s \geq 0\}$ are statistically independent, proving that the Poisson counting process defined by

$$N(t) = \sum_{n=1}^{\infty} u(t - T[n])$$

has independent increments.

(2) Since τ' is also exponentially distributed with the same parameter as $\tau[n+1], \tau[n+2], \ldots$, we can see from Fig. 2, by moving the horizontal axis up n units and the vertical axis right s time units, that $N(t+s) - N(s)$ is just another Poisson process with identical steering parameter λ.

Remarks:
We can view Poisson counting process from the following two perspectives:

(i) The Poisson process defined by

$$N(t) = \sum_{n=1}^{\infty} u(t - T[n])$$

can lead us to the following results:

- $N(t)$ for a given t is a Poisson random variable.
- $N(t)$ has independent increments.
- $N(t_2) - N(t_1)$ for $t_2 > t_1$ also has a Poisson distribution.

(ii) On the other hand, if we adopt another definition of the Poisson process as in Definition 7.2-2 of textbook, we can reach the result that the inter-arrival times are i.i.d. exponential random variables.

3 Classification of Random Processes

(1) Let $X(t)$ and $Y(t)$ be two random processes.

(a) $X(t)$ and $Y(t)$ are ***uncorrelated*** if

$$E[X(t_1)Y^*(t_2)] = E[X(t_1)]E[Y^*(t_2)]$$

for all t_1 and t_2.

(b) $X(t)$ and $Y(t)$ are ***orthogonal*** if

$$E[X(t_1)Y^*(t_2)] = 0$$

for all t_1 and t_2.

(c) $X(t)$ and $Y(t)$ are ***independent*** if the nth order joint pdf of $X(t)$ and $Y(t)$ factors for all n.

(2) **Definition (Stationary)**

A random process $X(t)$ is ***stationary*** if its nth order joint distribution (or pdf) is the same as that of $X(t+T)$ for all T and for all order $n \geq 1$.

(3) **Definition (Wide-Sense Stationary)**

A random process $X(t)$ is ***wide-sense stationary (WSS)*** if its mean function is a constant, and

$$R_{XX}(t_1, t_2) = R_{XX}(t_1 + \tau, t_2 + \tau).$$

Or equivalently and more oftenly used, the autocorrelation function of the WSS random process $X(t)$ is written as

$$R_{XX}(\tau) = E[X(t + \tau)X^*(t)]$$

for all t and τ.

(4) Properties of $R_{XX}(\tau)$ for WSS $X(t)$:

(a) $|R_{XX}(\tau)| \leq R_X(0)$ for arbitrary τ.

(b) $|R_{XY}(\tau)|^2 \leq R_{XX}(0)R_{YY}(0)$ for WSS $X(t)$ and $Y(t)$.

(c) The sequence $R_{XX}(\tau)$ is complex-conjugate symmetric, i.e.

$$R_{XX}(\tau) = R^*_{XX}(-\tau).$$

(d) For all $N \geq 1$, all $t_1 < t_2 \ldots < t_N$ and all complex $a_1, \cdots, a_N$, we must have

$$\sum_{k=1}^{N} \sum_{l=1}^{N} a_k a_l^* R_{XX}(t_k - t_l) \geq 0.$$

4 Linear Systems

Consider the following bounded (stable) LTI system with an impulse response $h(t)$. Suppose the input random process $X(t)$ is WSS.

$$X(t) \rightarrow \boxed{\quad h(t) \quad} \rightarrow Y(t)$$

Then,

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$$

is also WSS. This can be seen by finding the mean and autocorrelation function of $Y(t)$.

1. The mean function

$$\begin{aligned} \mu_Y(t) &= \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)\mu_X d\tau \\ &= \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau, \end{aligned}$$

which is also a constant for a stable LTI system $h(t)$.

2. We need the cross-correlation function $R_{YX}(t_1, t_2)$ in order to find the autocorrelation function $R_{YY}(t_1, t_2)$ of the output random process $Y(t)$.

$$\begin{aligned} R_{YX}(t_1, t_2) &= E[X(t_1)Y^*(t_2)] \\ &= \int_{-\infty}^{\infty} h(\alpha)E\left[X(t_1-\alpha)X^*(t_2)\right]d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha)R_{XX}(t_1-t_2-\alpha)d\alpha \\ &= h(\tau) * R_{XX}(\tau) \quad (\tau \triangleq t_1 - t_2) \end{aligned}$$

We can similarly proceed to find that

$$R_{YY}(t_1, t_2) = h^*(-\tau) * R_{YX}(\tau) = h^*(-\tau) * h(\tau) * R_{XX}(\tau).$$

We see that $R_{YY}(t_1, t_2)$ depends only on the difference $\tau = t_1 - t_2$ of t_1 and t_2. Thus, we can conclude that $Y(t)$ is WSS. In general, we write

$$\begin{aligned} R_{YY}(\tau) &= E[Y(t+\tau)Y^*(t)] \\ &= h^*(-\tau) * h(\tau) * R_{XX}(\tau) \end{aligned}$$

for WSS $X(t)$ and stable LTI system $h(t)$.

When converting to frequency domain, we have

$$S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega),$$

where

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-j\omega\tau} d\tau$$

is the continuous time Fourier transform of $R_{YY}(\tau)$ and likewise for $H(\omega)$ as well as $S_{XX}(\omega)$.

$S_{XX}(\omega)$ is also defined to be the power spectral density of the WSS random process $X(t)$.

5 Power Spectral Density

This part is analogous to that described in topic 6 for random sequences. We begin with the definitions of power and energy signals according to the theory of signals and systems.

(1) Deterministic signals can be classified into (i) power signals, and (ii) energy signals.
(i) For power signal, the power of a signal $x(t)$ is defined by

$$\mathrm{P} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{1}$$

(ii) For energy signal, the energy of a signal $x(t)$ is defined by

$$\mathrm{E} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

The Parseval's theorem states the following relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega, \tag{2}$$

where the continuous time Fourier transform is defined by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt,$$

and, by the definition of a density, we can take $|X(\omega)|^2$ as the ***energy spectral density***.

(2) For a power signal $x(t)$, its energy is infinity. We can define a truncated version of $x(t)$ as

$$x_T(t) = \begin{cases} x(t) & -T \le t \le T \\ 0 & \text{otherwise,} \end{cases}$$

which has a finite energy

$$\begin{aligned} \mathrm{E} &= \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-T}^{T} |x(t)|^2 dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega, \end{aligned} \tag{3}$$

where the last equality comes from the Parseval's relation and $X_T(\omega)$ is the Fourier transform of $x_T(t)$.

By dividing equation (3) by $2T$ and taking the limit, we have

$$\begin{aligned} \mathrm{P} &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} |X_T(\omega)|^2 d\omega. \end{aligned}$$

We can see from the above that the power spectral density of the signal $x(t)$ is

$$\mathrm{PSD} = \lim_{T\to\infty} \frac{1}{2T} |X_T(\omega)|^2,$$

under which the area is the signal power.

(3) For a random process $X(t)$, the sample function $X(t, \varepsilon_i)$ for each outcome $\varepsilon_i \in \Omega$ is deterministic and can be plugged into the above relation. That is, the PSD for $X(t, \varepsilon_i)$ is

$$\lim_{T\to\infty} \frac{1}{2T} |X_T(\omega, \varepsilon_i)|^2,$$

where $X_T(\omega, \varepsilon_i)$ is the Fourier transform of the similarly truncated $X_T(t, \varepsilon_i)$.

By averaging all realizations of sample functions, we have the ***average power***

$$\begin{aligned} \mathrm{P}_{\mathrm{ave}} &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[|X(t, \varepsilon_i)|^2\right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(\omega, \varepsilon_i)|^2\right] d\omega. \end{aligned}$$

So, the power spectral density $S_{XX}(\omega)$ of the random process $X(t)$ that bears more physical meanings is from the above

$$\boxed{S_{XX}(\omega) = \lim_{T\to\infty} \tfrac{1}{2T} E\left[|X_T(\omega, \varepsilon_i)|^2\right]}, \tag{4}$$

which is how equation (7.5-9b) in the textbook comes from. Next, we will show that the equation (4) is indeed the Fourier transform of the correlation function of $R_{XX}(\tau)$.

(4) (**Wiener-Khinchine Theorem**)

The PSD, if it exists, of a random process $X(t)$ is the Fourier transform of $R_{XX}(\tau)$. That is

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(\omega,\varepsilon_i)|^2\right]$$

and $R_{XX}(\tau)$ are continuous time Fourier transform pair.

Proof:

$$\begin{aligned}
\frac{1}{2T} E\left[|X_T(\omega,\varepsilon_i)|^2\right] &= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1-t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2 \\
&= \int_{-2T}^{2T} R_{XX}(\tau)\left(1-\frac{|\tau|}{2T}\right)e^{-j\omega\tau}d\tau.
\end{aligned}$$

Then, it follows, by taking the limit of the above,

$$\begin{aligned}
S_{XX}(\omega) &= \lim_{T\to\infty}\frac{1}{2T} E\left[|X_T(\omega,\varepsilon_i)|^2\right] \\
&= \lim_{T\to\infty}\int_{-2T}^{2T} R_{XX}(\tau)\left(1-\frac{|\tau|}{2T}\right)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau.
\end{aligned}$$

■

(5) Properties of PSD $S_{XX}(\omega)$:

- $S_{XX}(\omega)$ is real.
- $S_{XX}(\omega)$ is an even function if $X(t)$ is real.
- $S_{XX}(\omega) \geq 0$ for all ω.

6 Markov Processes

(1) **Definition (Markov Property)**

— If the pdf of a continuous-valued random process $X(t)$ has the property

$$f_X\left(x_n \Big| x_{n-1}, x_{n-2}, \cdots, x_1; t_n, \ldots, t_1\right) = f_X\left(x_n \Big| x_{n-1}; t_n, t_{n-1}\right)$$

for all $x_1, \cdots, x_n$ and all $t_n > t_{n-1} \cdots > t_1$ with $n > 0$, then we say $X(t)$ is a continuous-valued ***Markov process***.

— If the pmf of a discrete-valued random process $X(t)$ has the property

$$P_X\left(x_n \Big| x_{n-1}, x_{n-2}, \cdots, x_1; t_n, \ldots, t_1\right) = P_X\left(x_n \Big| x_{n-1}; t_n, t_{n-1}\right)$$

for all $x_1, \cdots, x_n$ and all $t_n > t_{n-1} \cdots > t_1$ with $n > 0$, then we say $X(t)$ is a ***discrete-valued*** Markov process, or a **Markov chain** if $X(t)$ takes on a set of finite or countable discrete values.

(2) Any independent increment process is Markov.

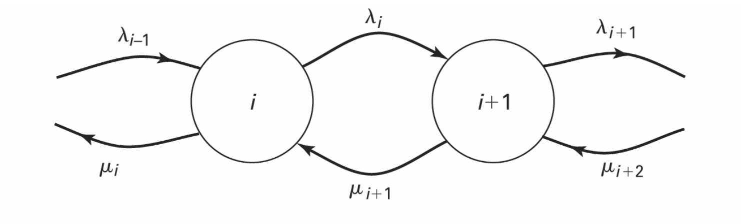

Figure 5: State transition diagram for an M/M/1 queue.

(3) **M/M/1 Queue**

M/M/1 queue is the simplest model in ***Queueing Theory***. The first two M's respectively stands for Markovian and Memoryless. The memoryless property means the exponential distribution is involved in the M/M/1 queue. The 1 stands for that there is one server (processer) in the queueing system. Summarizing, an M/M/1 queue has

— Poisson arrivals (or equivalently, exponential inter-arrival time)

— Exponential service time

— Single server

— An infinite length buffer

Let's see a specific example.

The number $N(t)$ of packets in an infinite size buffer can be modeled by an M/M/1 queue. The waiting time W_n for the nth packet in the queue can be described by

$$W_n = \max\Big\{0, W_{n-1} + \tau_s[n-1] - \tau[n]\Big\},$$

where

$\tau_s[n-1]$: service time to process the $(n-1)$th packet in the buffer

$\tau[n]$: inter-arrival time between the $(n-1)$th and nth packet

The inter-arrival time $\tau[n]$ and the service time $\tau_s[n]$ are statistically independent and follow exponential distributions with parameters λ_n and μ_n, respectively. The number $N(t)$ of packets in the M/M/1 queue at any time t can be visualized by a Markov chain state transition diagram as shown in Fig. 5.

What are the state probabilities $P[N(t) = j] \triangleq P_j$ in the ***steady state***?

We begin with the ***transition probability***

$$P[N(t + \triangle t) = j|N(t) = i],$$

by which we aim at finding

$$P[N(t + \triangle t) = j] = \sum_i P[N(t + \triangle t) = j|N(t) = i] \cdot P[N(t) = i]. \quad (5)$$

The transition probabilities are nonzero only when $i = j-1, j$ and $j+1$ during small amount of time increase $\triangle t$.

$$\begin{aligned}
P\Big[N(t + \triangle t) = j\Big|N(t) = j-1\Big] &= P\Big[0 < \tau[j] \leq \triangle t \ \text{ and } \ \tau_s[j-1] > \triangle t\Big] \\
P\Big[N(t + \triangle t) = j\Big|N(t) = j+1\Big] &= P\Big[0 < \tau_s[j+1] \leq \triangle t \ \text{ and } \ \tau[j+1] > \triangle t\Big] \\
P\Big[N(t + \triangle t) = j\Big|N(t) = j\Big] &= P\Big[\tau[j] > \triangle t \ \text{ and } \ \tau_s[j] > \triangle t\Big].
\end{aligned}$$

By carrying out the above, we reach

$$\begin{aligned}
P[N(t + \triangle t) = j|N(t) = j-1] &= (1 - e^{-\lambda \triangle t})e^{-\mu \triangle t} = \lambda \triangle t + o(\triangle t) \\
P[N(t + \triangle t) = j|N(t) = j+1] &= (1 - e^{-\mu \triangle t})e^{-\lambda \triangle t} = \mu \triangle t + o(\triangle t) \\
P[N(t + \triangle t) = j|N(t) = j] &= e^{-(\lambda+\mu)\triangle t} = 1 - (\lambda + \mu)\triangle t + o(\triangle t),
\end{aligned}$$

where $o(t)$ satisfies $\lim_{\triangle t \to 0} \frac{o(t)}{\triangle t} = 0$. Writing equation (5) in a matrix form for all j and i, we have

$$\mathbf{p}(t + \triangle t) = \mathbf{B} \cdot \mathbf{p}(t) + \mathbf{o}(t), \quad (6)$$

where $\mathbf{p}(t) = \big[P[N(t) = 0], \cdots, P[N(t) = j], \cdots\big]^T$ and

$$\mathbf{B} = \begin{bmatrix} 1 - \lambda \triangle t & \mu \triangle t & 0 & \cdots & \\ \lambda \triangle t & 1 - (\lambda + \mu)\triangle t & \mu \triangle t & 0 & \cdots \\ 0 & \lambda \triangle t & 1 - (\lambda + \mu)\triangle t & \mu \triangle t & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Rearranging equation (6), we get

$$\frac{d\mathbf{p}(t)}{dt} = \mathbf{A} \cdot \mathbf{p}(t),$$

where

$$\mathbf{A} = \begin{bmatrix} -\lambda & \mu & 0 & \cdots & \\ \lambda & -(\lambda+\mu) & \mu & 0 & \cdots \\ 0 & \lambda & -(\lambda+\mu) & \mu & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

In the steady state, the derivative is a zero vector $d\mathbf{P}(t)/dt = \mathbf{0}$, giving

$$\mathbf{A} \cdot \mathbf{p} = \mathbf{0}.$$

Therefore, with $\sum_{j=1}^{\infty} P_j = 1$, we have the steady-state state probabilities

$$P_j = \rho^j(1-\rho),$$

where $\rho = \lambda/\mu$ and we have assumed $\rho < 1$ for convergence.

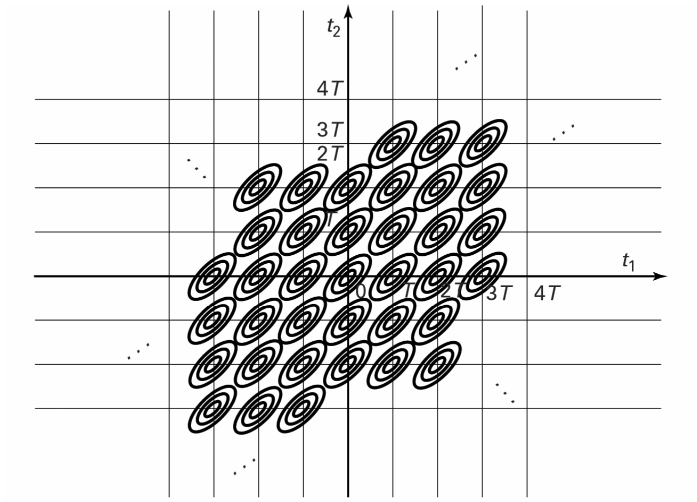

Figure 7.6-1
Possible contours of the covariance function of a wide-sense (WS) periodic random process.

Figure 6: Contour of a wide-sense periodic process.

7 Cyclostationary Process

(1) **Periodic**

A random process $X(t)$ is ***wide-sense periodic*** if there is a $T > 0$ such that

$$\mu_X(t) = \mu_X(t+T)$$

and

$$K_{XX}(t_1, t_2) = K_{XX}(t_1 + T, t_2) = K_{XX}(t_1, t_2 + T)$$

for all t, t_1, and t_2.

Example:
The random process

$$X(t) = \sum_{k=1}^{\infty} A_k \exp\left(j\frac{2\pi k t}{T}\right)$$

with A_k being random variables are wide-sense periodic.

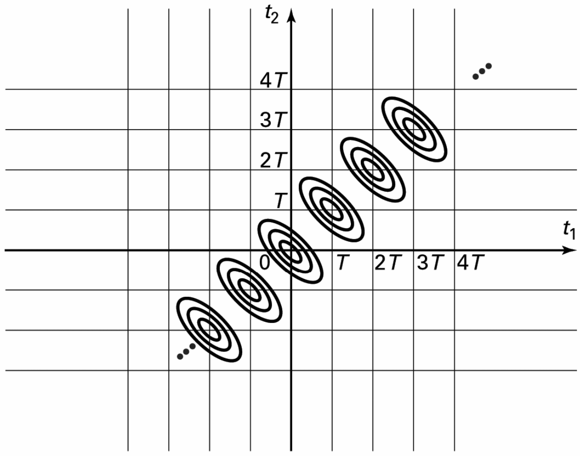

Figure 7.6-2

Possible contour plot of covariance function of WS cyclostationary random process.

Figure 7: Contour of a wide-sense cyclostationary process.

(2) **Cyclostationary**

A random process $X(t)$ is ***wide-sense cyclostationary*** if there exists a positive value T such that

$$\mu_X(t) = \mu_X(t+T)$$

and

$$K_{XX}(t_1, t_2) = K_{XX}(t_1+T, t_2+T)$$

for all t, t_1, and t_2.

Example:

The binary phase-shift keying (BPSK) random process

$$X(t) = \sum_{k=-\infty}^{\infty} \cos\Big(2\pi f(t-kT) + \theta[k]\Big) p(t-kT),$$

with

$$\theta[k] = \begin{cases} \pi/2 & p = 1/2 \\ -\pi/2 & p = 1/2 \end{cases}$$

and $p(t) = u(t) - u(t-T)$, is wide-sense cyclostationary.