Signal and Linear System Analysis

Carrson C. Fung
Dept. of Electronics Engineering
National Chiao Tung University

Signal Model and Classifications

- Deterministic signals
  - Completely specified function of time: predictable, no uncertainty. E.g.

$$x(t) = A\cos\omega_0 t, \quad -\infty < t < \infty,$$

where A and ω_0 are constants

- Random/Stochastic signals
  - Take on random values at any given time instant and characterized by pdf: not completely predictable, with uncertainty. E.g. $x(n)$ = value of a die shown when tossed at time index n
  - If the signal is random, how do we describe (model) it?

Signal Model and Classifications

- Periodic signal
  - A signal $x(t)$ is periodic iff there exists a constant T_0, such that $x(t + T_0) = x(t), \forall t$. The smallest such T_0 is called fundamental period or simply period
- Aperiodic signal
  - Cannot find a finite T_0 such that $x(t + T_0) = x(t), \forall t$

Signal Model and Classifications

- Phasor signal and spectra
  - A special periodic function

$$\tilde{x}(t) = Ae^{j(\omega_0 t+\theta)} = Ae^{j\theta}e^{j\omega_0 t},$$

$$\tilde{x}(t) \triangleq \text{rotating phasor}, \quad Ae^{j\theta} \triangleq \text{phasor}, \quad A,\theta \in \mathbb{R}$$

  - Why use this complex signal?
    - Key part of modulation theory
    - Construction signal for almost any signal
    - Easy mathematical analysis for signal
    - Phase carries time delay information

Signal Model and Classifications

- More on phasor signal
  - Information is contained in A and t (given a fixed f_0 or ω_0)
  - The related real sinusoidal function

$$x(t) = A\cos(\omega_0 t + \theta) = \text{Re}\{\tilde{x}(t)\}, \text{ or } x(t) = A\cos(\omega_0 t + \theta) = \frac{1}{2}\tilde{x}(t) + \frac{1}{2}\tilde{x}^*(t)$$

  - In vector form graphically

$$\tilde{x}(t) = Ae^{j(\omega_0 t + \theta)}$$

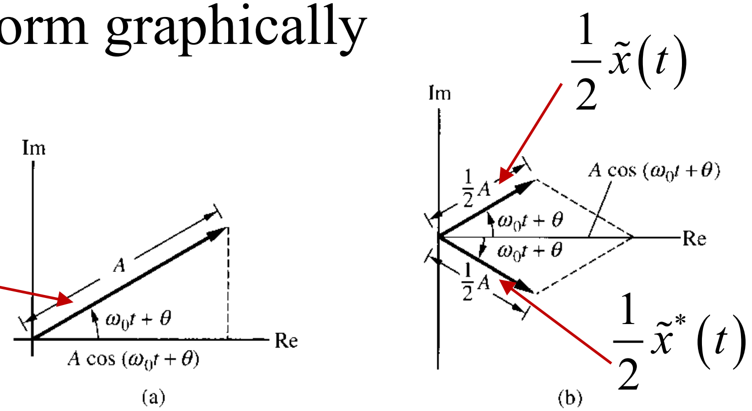


Figure 2.2
Two ways of relating a phasor signal to a sinusoidal signal. (a) Projection of a rotating phasor onto the real axis. (b) Addition of complex conjugate rotating phasors.

Signal Model and Classifications

- Frequency-domain representation
  - Line spectra

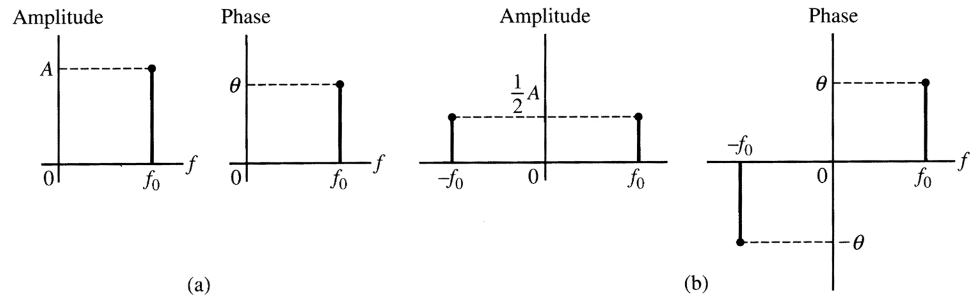


Figure 2.3
Amplitude and phase spectra for the signal $A\cos(\omega_0 t + \theta)$. (a) Single sided. (b) Double sided.

$\tilde{x}(t)$ $x(t)$

Single-sided (SS) amplitude and phasor vs. double-sided (DS):

Signal Model and Classifications

- Singular functions

Unit impulse function: $\delta(t)$

1. Definition

$$\int_t \delta(t)\ dt = 1$$

$$\Rightarrow \int_t x(t)\delta(t)\ dt = x(0)\int_t \delta(t)\ dt = x(0)$$

2. Sifting property:

Define a precise sample point of $x(t)$ at time t (or t_0 if $\delta(t - t_0)$)

$$x(t_0) = \int_t x(t)\delta(t - t_0)\ dt$$

3. Basic function for linearly constructing a time signal

$$x(t) = \int_\tau x(\tau)\delta(t - \tau)\ d\tau$$

4. Some properties

$$\delta(at) = \frac{1}{|a|}\delta(t); \quad \delta(t) = \delta(-t): \text{ even function}$$

Signal Model and Classifications

5. What is $\delta(t)$ precisely? Some intuitive ways of realizing it:

E.g. 1 $$\delta(t)=\begin{cases}\lim_{\varepsilon\to 0}\dfrac{1}{2\varepsilon}, & |t|<\varepsilon,\\ 0, & \text{otherwise}\end{cases}$$

E.g. 2 $$\delta(t)=\lim_{\varepsilon\to 0}\varepsilon\left(\frac{1}{\pi t}\sin\frac{\pi t}{\varepsilon}\right)^2$$

$\Rightarrow$ Any signal having unit area and zero width in the limit as some parameter goes to zero is a suitable representation

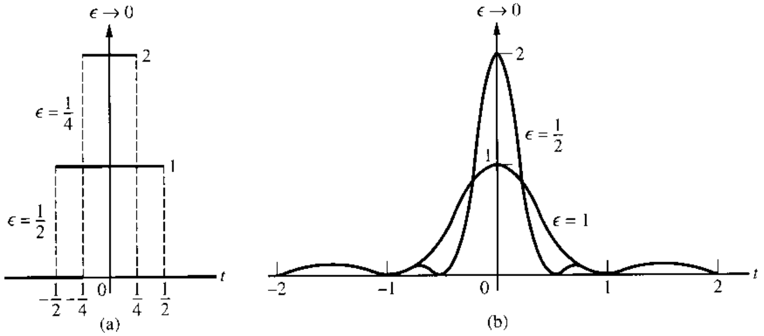


Figure 2.4
Two representations for the unit impulse function in the limit as $\epsilon \to 0$. (a) $(1/2\epsilon)\Pi(t/2\epsilon)$. (b) $\epsilon[(1/\pi t)\sin(\pi t/\epsilon)]^2$.

Signal Model and Classifications

Unit step function: $u(t)$

Definition

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\ d\lambda; \quad \delta(t) = \frac{du(t)}{dt}$$

Signal Classifications: Energy & Power

This classification will be needed for the later analysis of deterministic and random signals

Energy: $E \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt$

Power: $P \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$

Energy Signals: iff $0 < E < \infty \;\; (P = 0)$

Power Signals: iff $0 < P < \infty \;\; (E = \infty)$

Signal Classifications: Energy & Power

Example 1:

$$x_1(t) = Ae^{-\alpha t}u(t), \quad \text{for } \alpha > 0$$

$$E = \lim_{T\to\infty} \int_{-T}^{T} \left| Ae^{-\alpha t}u(t) \right|^2 dt$$

$$= A^2 \lim_{T\to\infty} \int_{0}^{T} e^{-2\alpha t} dt$$

$$= -A^2 \lim_{T\to\infty} \frac{1}{2\alpha}\left[e^{-2\alpha T} - 1 \right]$$

$$= \frac{A^2}{2\alpha} \qquad \Rightarrow \text{energy signal}$$

Signal Classifications: Energy & Power

Example 2:

$$x_2(t) = Au(t)$$

$$\begin{aligned} E &= \lim_{T\to\infty} \int_{-T}^{T} \left|Au(t)\right|^2 dt \\ &= A^2 \lim_{T\to\infty} T \\ &= \infty \end{aligned}$$

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|Au(t)\right|^2 dt \\ &= \frac{A^2}{2} \qquad \Rightarrow \text{power signal} \end{aligned}$$

Signal Classifications: Energy & Power

Example 3:

$$x_3(t) = A\cos(\omega_0 t + \theta)$$

$$E = \lim_{T\to\infty}\int_{-T}^{T}\left|A\cos(\omega_0 t + \theta)\right|^2 dt$$

$$= A^2 \lim_{T\to\infty}\int_{t_0}^{t_0+T}\cos^2(\omega_0 t + \theta)\,dt$$

$$= A^2 \lim_{T\to\infty}\int_{t_0}^{t_0+T}\frac{1}{2}dt + \underbrace{\int_{t_0}^{t_0+T}\frac{1}{2}\cos\left[2(\omega_0 t + \theta)\right]dt}_{=0}$$

$$= A^2 \lim_{T\to\infty}\int_{t_0}^{t_0+T}\frac{1}{2}dt = \infty$$

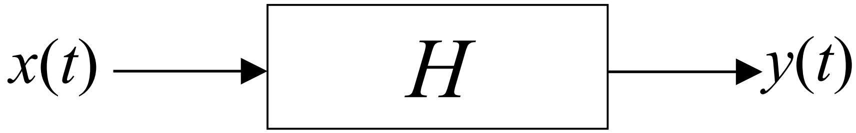
$$P = \lim_{T\to\infty}\frac{A^2}{2T}\int_{t_0}^{t_0+T}\frac{1}{2}dt$$

$$= \frac{A^2}{4} \qquad \Rightarrow \text{power signal}$$

Signal Classifications: Energy & Power

- If $x(t)$ is periodic, then it is meaningless to find its energy, we only need to check its power
- Noise is often persistent and is often a power signal
- A realizable LTI system can be represented by a signal and mostly is an energy signal
- Power measure is useful for signal and noise analysis
- The energy and power classifications of signals are mutually exclusive, i.e. cannot be both at the same time. But a signal can be neither energy nor power signal

Signals and Linear Systems

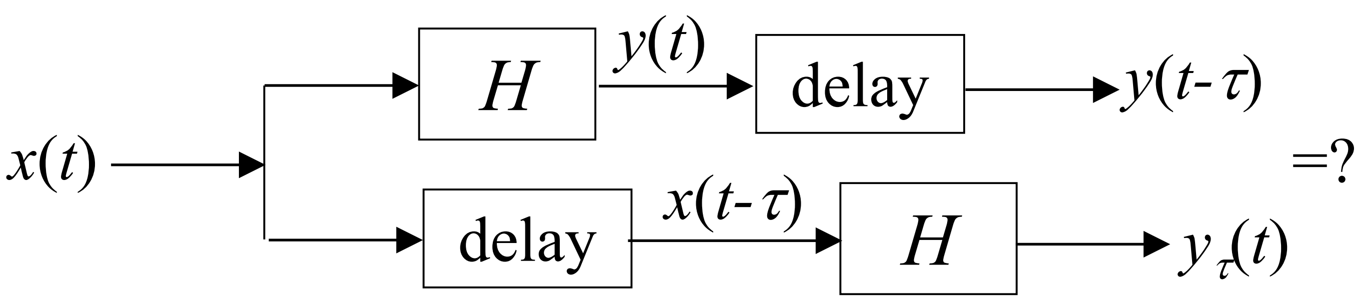


Linear and Time-Invariant (LTI) System

Linear System satisfies superposition principle:

$$y(t) = H\left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right] = H\alpha_1 x_1(t) + H\alpha_2 x_2(t) = y_1(t) + y_2(t)$$

Time-Invariant: $y(t) = H\left[x(t)\right] \Rightarrow H\left[x(t-\tau)\right] = y(t-\tau)$



LTI Examples

1) $y(t) = x(\alpha t)$

2) $y(t) = x(t) + c$, where c is a constant

Complete Characterization of LTI Systems

The unit impulse function is key to the characterization

$$h(t) \triangleq H\left[\delta(t)\right]$$

$$x(t) = \int_\lambda x(\lambda)\delta(t-\lambda)\,d\lambda$$

$$\begin{aligned} y(t) &= H\left[x(t)\right] \\ &= H\left[\int_\lambda x(\lambda)\delta(t-\lambda)\,d\lambda\right] \\ &= \int_\lambda x(\lambda) H\left[\delta(t-\lambda)\right] d\lambda \end{aligned}$$

If TI:

$$\begin{aligned} &= \int_\lambda x(\lambda) h(t-\lambda)\,d\lambda \\ &= x(t) * h(t) \qquad \text{(convolution)} \end{aligned}$$

Convolution Example

$$x(t) = e^{-\alpha t}u(t), \quad \text{for } \alpha > 1, \; h(t) = u(t) - u(t-2)$$

Find $y(t)$.

Convolution Using Matrices and Vectors (Digression on DT convolution)

Suppose $L_h = 3$ and $L_x = 4$:

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ 0 & h[2] & h[1] & h[0] \\ 0 & 0 & h[2] & h[1] \\ 0 & 0 & 0 & h[2] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} \Leftrightarrow \mathbf{y} = \mathbf{H}\mathbf{x}$$

Suppose we are only interested in a single output sample,

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \mathbf{h}^T \mathbf{x}(n),$$

where

$$\mathbf{h} = \begin{bmatrix} h[0] & h[1] & \cdots & h[L_h - 1] \end{bmatrix}^T$$

$$\mathbf{x}(n) = \begin{bmatrix} x[n] & x[n-1] & \cdots & x[n - L_h + 1] \end{bmatrix}^T$$

BIBO Stability

- BIBO Stability
  - If and only if every bounded input sequence produces a bounded output sequence.

$$
\begin{aligned}
\max\left|y(t)\right| &= \max\left|\int_{\lambda} x(\lambda)h(t-\lambda)\,d\lambda\right| \\
&\le \max\left|x(t)\right|\int_{\lambda}\left|h(\lambda)\right|d\lambda \\
&< \infty
\end{aligned}
$$

$$\Rightarrow \text{System is BIBO iff } \int_{\lambda}\left|h(\lambda)\right|d\lambda < \infty$$

BIBO Stability Examples

1) $y(t) = tx(t)$

Given $|x(t)| < B_x$. Then

$$|T(x(t))| = |tx(t)| \leq |t||x(t)| = |t| B_x$$

As $t \to \infty$, $|T(x[n])|$ will be unbounded, therefore, the system is not BIBO stable

2) $y(t) = x(\alpha t)$, for $\alpha > 1$

Given $|x(t)| < B_x$. Then

$$|T(x(t))| = |x(\alpha t)| < B_x$$

The inequality is true because $x(\alpha t)$ is a compressed version of $x(t)$. Therefore, the system is BIBO stable

Causality

- A system is causal if current output does not depend on future input, or current input does not contribute to the output in the past

$$y(t)=\int_{-\infty}^{\infty}h(\lambda)x(t-\lambda)\,d\lambda=\int_{0}^{\infty}h(\lambda)x(t-\lambda)\,d\lambda$$

$$\Rightarrow h(t)=0,\quad \text{for}\quad t<0$$

Eigenfunctions of LTI System

Consider

Let $x(t) = Ae^{jst}$, where $s = \sigma + j\omega$. What is $y(t)$?

What is the relation to eigenvalues and eigenvectors?

System Transmission Distortion and System Frequency Response

- Since almost any input $x(t)$ can be represented by a linear combination of orthogonal sinusoidal basis functions $e^{j2\pi ft}$, we only need to inject $Ae^{j2\pi ft}$ to the system to characterize the system's properties, and the eigenvalue

$$\alpha = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt = H(f)$$

carries all the system information responding to $Ae^{j2\pi ft}$

- In communication theory, transmission distortion is of primary concern in high-quality transmission of data. Hence, the proper representation for the transmission channel (remember, convolutive noise is troublesome)

3 Types of Distortion of a Channel

- Amplitude distortion
  - Linear system but the amplitude response is not constant
- Phase distortion
  - Phase (delay) distortion
    - Linear system but the phase shift is not a linear function of frequency
- Nonlinear distortion
  - Nonlinear system

Fourier Series

Complex exponential respresentation

$$x(t) = \sum_n X_n e^{j2\pi n f_0 t}$$

$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-j2\pi n f_0 t} dt$$

Sinusoidal respresentation

$$x(t) = X_0 + \sum_{n=1}^{\infty}\left[X_n e^{j2\pi n f_0 t} + X_{-n} e^{j2\pi(-n) f_0 t}\right]$$

Phasor representation: $X_n = |X_n| e^{j\angle X_n}$

if $x(t)$ is real

$$X_{-n} = |X_{-n}| e^{j\angle X_{-n}} = |X_n| e^{-j\angle X_n}$$

$$x(t) = X_0 + \sum_{n=1}^{\infty} |X_n|\left[e^{j(2\pi n f_0 t + \angle X_n)} + e^{-j(2\pi n f_0 t + \angle X_n)}\right]$$

$$= X_0 + \sum_{n=1}^{\infty} 2|X_n| \cos(2\pi n f_0 t + \angle X_n)$$

Note:

- $n = 1$ term is called the **fundamental**
- $n = 2, 3, \ldots$ terms are called the **2nd, 3rd , …harmonics**, respectively

Fourier Series

$$x(t) = X_0 + \sum_{n=1}^{\infty} 2|X_n|\cos(2\pi n f_0 t + \angle X_n)$$

Trigonometric respresentation

$$x(t) = X_0 + \sum_{n=1}^{\infty} 2|X_n|\left[\cos(\angle X_n)\cos(2\pi n f_0 t) - \sin(\angle X_n)\sin(2\pi n f_0 t)\right]$$

$$= X_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t),$$

where $a_n = 2|X_n|\cos(\angle X_n)$, $b_n = -2|X_n|\sin(\angle X_n)$ or

$$a_n = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(2\pi n f_0 t)\,dt, \quad b_n = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(2\pi n f_0 t)\,dt$$

Properties of Fourier Series

$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-j2\pi n f_0 t}dt$$

"DC" coefficient: $X_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,dt =$ average value of $x(t)$

"AC" coefficients: $X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-j2\pi n f_0 t}dt$

$$= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(2\pi n f_0 t)\,dt - j\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(2\pi n f_0 t)\,dt$$

If $x(t)$ is real, then $X_n = \mathrm{Re}\{X_n\} + j\,\mathrm{Im}\{X_n\} = |X_n|e^{j\angle X_n}$,

where $\mathrm{Re}\{X_n\} = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(2\pi n f_0 t)\,dt$

$$\mathrm{Im}\{X_n\} = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(2\pi n f_0 t)\,dt$$

Hence

$$\mathrm{Re}\{X_n\} = \mathrm{Re}\{X_{-n}\}, \quad \mathrm{Im}\{X_n\} = -\mathrm{Im}\{X_{-n}\}$$

$X_{-n} = X_n^* \quad \rightarrow \quad |X_n| = |X_{-n}|$ (even function),

$\angle X_n = -\angle X_{-n}$ (odd function)

Properties of Fourier Series

Linearity: $x(t) \leftrightarrow a_k,\ \ y(t) \leftrightarrow b_k$

$$Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$$

Time reversal: $x(t) \leftrightarrow a_k$

$$x(-t) \leftrightarrow a_{-k}$$

Time & freq shifting: $x(t - t_0) \leftrightarrow e^{-j2\pi k f_0 t_0} a_k;\ \ e^{jM2\pi f_0 t} x(t) \leftrightarrow a_{k-M}$

Time scaling $x(\alpha t), \alpha > 0$ (periodic with $\frac{T_0}{\alpha}$) $\leftrightarrow a_k$

Multiplication: $x(t) \leftrightarrow a_k,\ \ y(t) \leftrightarrow b_k$

$$x(t)y(t) \leftrightarrow \sum_{\ell} a_{\ell} b_{k-\ell}$$

Conjugation and Conjugate Symmetry

$$x(t) \leftrightarrow a_k$$

$$x^*(t) \leftrightarrow a_{-k}^*$$

if $x(t)$ is real $\Rightarrow\ a_{-k} = a_k^*$

Properties of Fourier Series

Parseval's Theorem:

Power in time domain = power in frequency domain

$$
\begin{aligned}
P_x &= \frac{1}{T_0}\int_{T_0} \left|x(t)\right|^2 dt \\
&= \frac{1}{T_0}\int_{T_0} \sum_{m,n} X_m X_n^* e^{j2\pi(m-n)f_0 t} dt \\
&= \frac{1}{T_0}\sum_{m,n} X_m X_n^* \int_{T_0} e^{j2\pi(m-n)f_0 t} dt \\
&= \frac{1}{T_0}\sum_{m,n} X_m X_n^* \int_{T_0} \cos\left(2\pi\left(m-n\right)f_0 t\right) + j\sin\left(2\pi\left(m-n\right)f_0 t\right) dt \\
&= \begin{cases} \frac{1}{T_0} T_0 \sum_n \left|X_n\right|^2, & m = n, \\ 0, & m \neq n \end{cases} \\
&= \sum_n \left|X_n\right|^2
\end{aligned}
$$

Fourier Series for Several Periodic Signals

Table 2.1 Fourier Series for Several Periodic Signals

Signal (one period)	Coefficients for exponential Fourier series
1. Asymmetrical pulse train; period $= T_0$:	
$x(t) = A\Pi\left(\frac{t-t_0}{\tau}\right), \tau < T_0$	$X_n = \frac{A\tau}{T_0}\text{sinc}(nf_0\tau)e^{-j2\pi nf_0 t_0}$
$x(t) = x(t+T_0)$, all t	$n = 0, \pm 1, \pm 2, \ldots$
2. Half-rectified sine wave; period $= T_0 = 2\pi/\omega_0$:	
$x(t) = \begin{cases} A\sin(\omega_0 t), & 0 \le t \le T_0/2 \\ 0, & -T_0/2 \le t \le 0 \end{cases}$	$X_n = \begin{cases} \frac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \cdots \\ 0, & n = \pm 3, \pm 5, \cdots \\ -\frac{1}{4}jnA, & n = \pm 1 \end{cases}$
$x(t) = x(t+T_0)$ all t	
3. Full-rectified sine wave; period $= T_0 = \pi/\omega_0$:	
$x(t) = A\|\sin(\omega_0 t)\|$	$X_n = \frac{2A}{\pi(1-4n^2)}, \quad n = 0, \pm 1, \pm 2, \ldots$
4. Triangular wave:	
$x(t) = \begin{cases} -\frac{4A}{T_0}t + A, & 0 \le t \le T_0/2 \\ \frac{4A}{T_0}t + A, & -T_0/2 \le t \le 0 \end{cases}$	$X_n = \begin{cases} \frac{4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$
$x(t) = x(t+T_0)$, all t	

Example 1

Half-rectified sine wave; period $= T_0 = 2\pi / \omega_0$

$$x(t) = \begin{cases} A\sin(\omega_0 t), & 0 \le t \le T_0/2, \\ 0, & -T_0/2 \le t \le 0 \end{cases}$$

Recall $X_n = |X_n| e^{j\angle X_n}$

$$|X_n| = \sqrt{X_n X_n^*} = \begin{cases} \sqrt{\dfrac{A^2}{\pi^2\left(1-2n^2+n^4\right)}}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 3, \pm 5, \ldots \\ \dfrac{1}{4} nA, & n = \pm 1 \end{cases}$$

$$\measuredangle X_n = \tan^{-1}\left(\frac{\mathrm{Im}\{X_n\}}{\mathrm{Re}\{X_n\}}\right)$$

Example 1

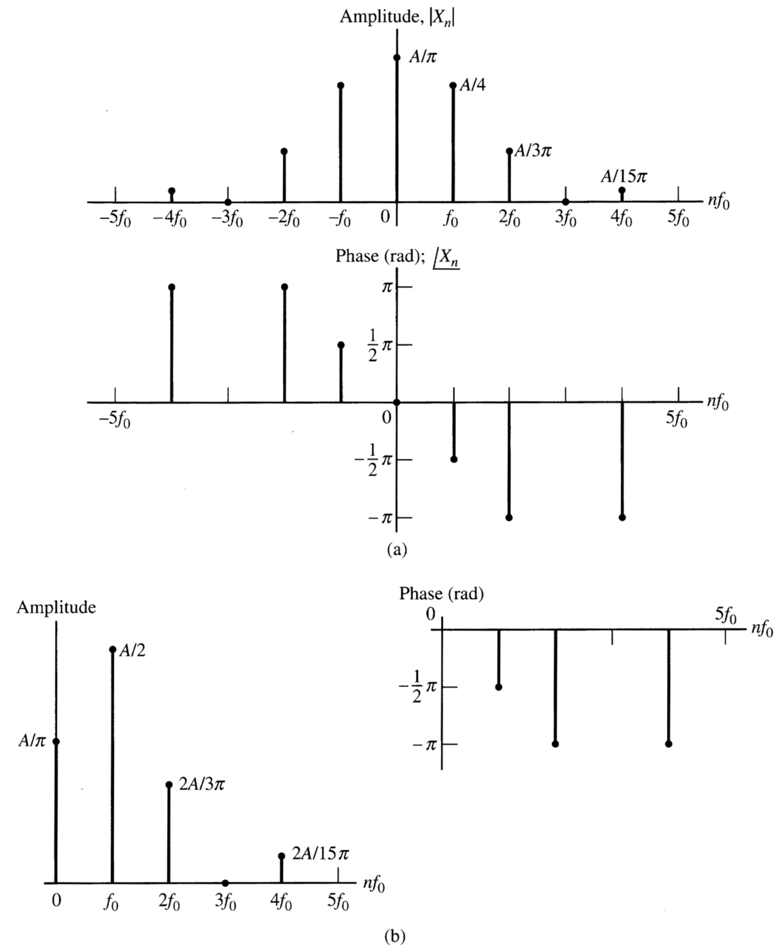


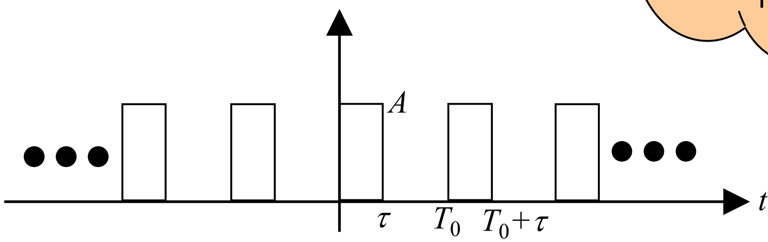
Figure 2.6
Line spectra for half-rectified sinewave. (a) Double sided. (b) Single sided.

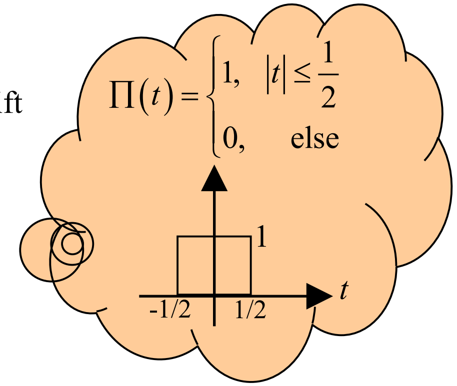
Example 2

Periodic extension

shift

$$x(t)=\sum_n A\Pi\left(\frac{t-nT_0-\frac{1}{2}\tau}{\tau}\right)$$





$$\Pi(t)=\begin{cases}1, & |t|\le\frac{1}{2}\\ 0, & \text{else}\end{cases}$$

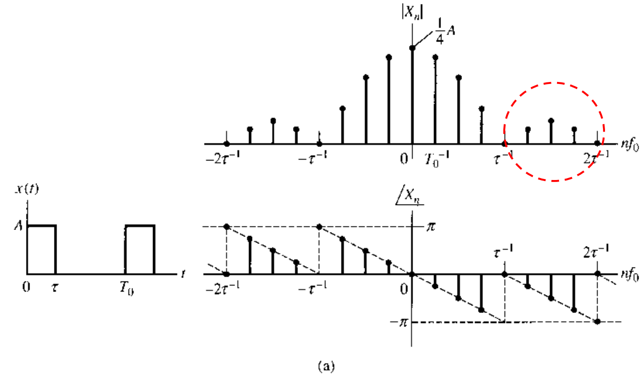


Figure 2.7
Spectra for a periodic pulse train signal. (a) $\tau=\frac{1}{4}T_0$. (b) $\tau=\frac{1}{8}T_0$; T_0 same as in (a). (c) $\tau=\frac{1}{8}T_0$; τ same as in (a).

$$X_n=\frac{A\tau}{T_0}\operatorname{sinc}(nf_0\tau)e^{-j\pi nf_0\tau}\qquad \left(\text{recall } \delta(t-\tau)\leftrightarrow\frac{1}{T_0}e^{-jn\omega_0\tau}\right)$$

$$\Rightarrow|X_n|=\frac{A\tau}{T_0}\left|\operatorname{sinc}(nf_0\tau)\right|,\quad \angle X_n=\begin{cases}-\pi nf_0\tau, & \operatorname{sinc}(nf_0\tau)>0,\\ -\pi nf_0\tau+\pi, & nf_0>0 \text{ and } \operatorname{sinc}(nf_0\tau)<0,\\ -\pi nf_0\tau-\pi, & nf_0<0 \text{ and } \operatorname{sinc}(nf_0\tau)<0,\end{cases}$$

$+\pi$ and $-\pi$ added to account for the fact that $|\operatorname{sinc}(nf_0\tau)|$ = -sinc($nf_0\tau$) when sinc($nf_0\tau$) < 0. Choice of + or – are arbitrary, as long as the phase function is odd

Example 2

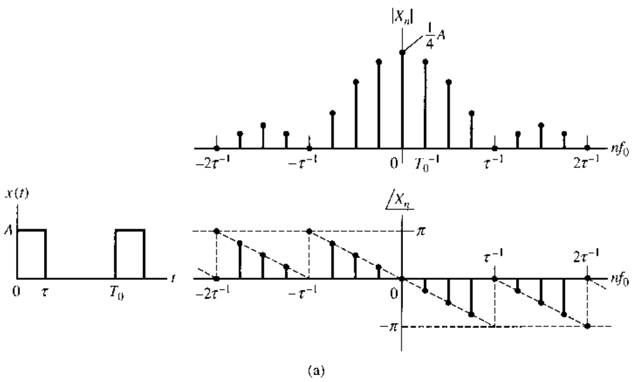


Figure 2.7
Spectra for a periodic pulse train signal. (a) $\tau = \frac{1}{4}T_0$. (b) $\tau = \frac{1}{8}T_0$; T_0 same as in (a). (c) $\tau = \frac{1}{8}T_0$; τ same as in (a).

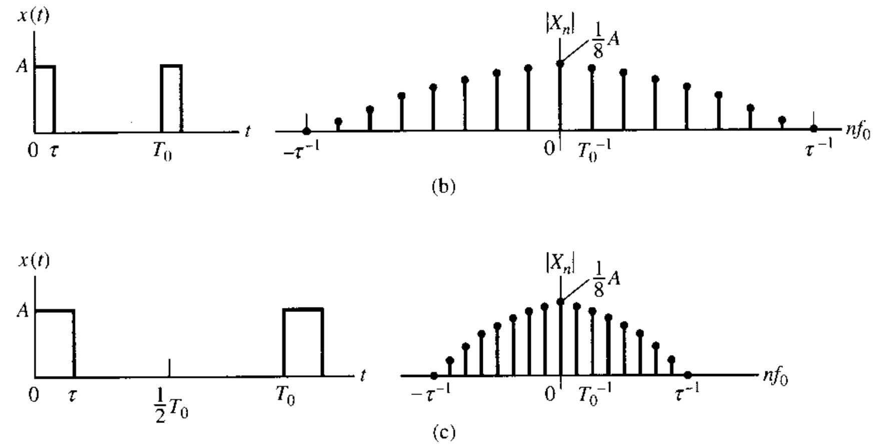


Figure 2.7
Continued.

$$X_n = \frac{A\tau}{T_0}\operatorname{sinc}\left(nf_0\tau\right)e^{-j\pi nf_0\tau}$$

- Zero-crossings occur at $1/\tau$ Hz
- τ decreases (increases), distance between zero-crossings increases (decreases)
- T_0 decreases (increases), space between the spectra lines increases (decreases)

Fourier Series and Fourier Transform

Fourier Series	Fourier Transform
Good orthogonal basis functions for a periodic function 1. Intuitively, basis functions should also be periodic 2. Intuitively, periods of the basis functions should be equal to the period or integer fractions of the target signal 3. Fourier found that sinusoidal functions are good and smooth functions to expand a periodic function	**Good orthogonal basis functions for an aperiodic function** 1. Already know sinusoidal functions are good choice 2. Sinusoidal components should not be in a "fundamental & harmonic" relationship 3. Aperiodic signals are mostly finite duration 4. Consider aperiodic function as a special case of periodic function with infinite period
Synthesis & analysis (reconstruction & projection) Given period $x(t)$ with period $T_0 = 1/f_0$, $\omega_0 = 2\pi f_0$, it can be synthesized as $x(t) \approx \sum_n X_n e^{jn\omega_0 t}$ X_n is the spectra coefficient, spectra amplitude response To synthesize, it must first analyze it and find X_n By orthogonality $X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t} dt$	**Synthesis & analysis (reconstruction & projection)** Given aperiod $x(t)$ with period $T_0 = 1/df \to \infty$, $\omega_0 = d\omega = 2\pi df$, it can be synthesized as $x(t) \approx \lim_{d\omega \to 0} \sum_n X_n e^{jnd\omega t} = \frac{1}{2\pi}\int_\omega X(\omega) e^{j\omega t} d\omega = \int_f X(f) e^{j2\pi ft} df$ By orthogonality $X(f) = \int_t x(t) e^{-j2\pi ft} dt$ (FT/freq response of $x(t)$) $\Rightarrow x(t) = \int_f X(f) e^{j2\pi ft} df$ (Inverse FT)

Fourier Series and Fourier Transform

Fourier Series	Fourier Transform
Frequency components 1. Decompose a periodic signal into countable frequency components 2. Has a fundamental frequency and many other harmonics $X_n e^{jn\omega_0 t} = \lvert X_n \rvert e^{j\angle X_n} e^{jn\omega_0 t}$ $\lvert X_n \rvert$: amplitude $\angle X_n$: phase of $\lvert X_n \rvert$ 3. Discrete line spectral	**Frequency components** 1. Decompose an aperiodic signal into uncountable frequency components 2. No fundamental frequency and contain all possible freqs $X(f)e^{j2\pi ft} = \lvert X(f)\rvert e^{j\angle X(f)} e^{j2\pi ft},$ $-\infty < f < \infty$ 3. Continuous spectral density
Power Spectral Density $\lvert X_n \rvert^2$ and by Parseval's theorem $P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \lvert x(t)\rvert^2 dt = \sum_n \lvert X_n \rvert^2$	**Energy Spectral Density** $\lvert X(f)\rvert^2$ and by Parseval's theorem $E = \int_t \lvert x(t)\rvert^2 dt = \int_f \lvert X(f)\rvert^2 df$

Continuous-Time Fourier Transform (CTFT) (Pictorial Depiction)

- Main idea: Decomposing a signal into its sinusoidal components

  Analysis: $X(f) = \int_t x(t) e^{-j2\pi ft} dt$

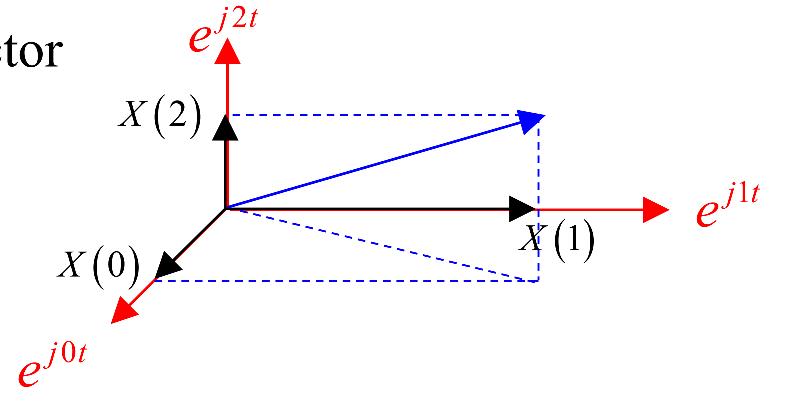
  Synthesis: $x(t) = \int_f X(f) e^{j2\pi ft} df$

  - A specific case of projection of vectors.
    - Sinusoidal/exponential functions (of different ω's) form the basis vectors.
    - Signal to be decomposed is the vector

Recall the projection matrix: $\mathbf{P} = \dfrac{\mathbf{v}\mathbf{v}^H}{\mathbf{v}^H\mathbf{v}}$.

If $\mathbf{v}$ is orthonormal(rewrite as $\mathbf{q}_i$), then

$$\mathbf{a} = \underbrace{\mathbf{q}_1 \underbrace{\mathbf{q}_1^H \mathbf{a}}_{\text{analysis}}}_{\text{synthesis}} + \underbrace{\mathbf{q}_2 \underbrace{\mathbf{q}_2^H \mathbf{a}}_{\text{analysis}}}_{\text{synthesis}} + \underbrace{\mathbf{q}_3 \underbrace{\mathbf{q}_3^H \mathbf{a}}_{\text{analysis}}}_{\text{synthesis}}$$



Symmetry Properties

For real aperiodic $x(t)$: $X_{-n} = X_n^*$, or $|X_n| = |X_{-n}|$, $\angle X_n = -\angle X_{-n}$

For real aperiodic $x(t)$: $X(f) = X^*(-f)$, or $|X(f)| = |X(-f)|$, $\angle X(f) = -\angle X(-f)$

Example 2.8:

Given $x(t) = A\Pi\left(\frac{t-t_0}{\tau}\right)$. Find the CTFT.

$$X(f) = \int_t A\Pi\left(\frac{t-t_0}{\tau}\right) e^{-j2\pi ft} dt$$

$$= A\int_{t_0-\tau/2}^{t_0+\tau/2} e^{-j2\pi ft} dt$$

$$= -\frac{A}{j2\pi f}\left[e^{-j2\pi ft}\right]_{t_0-\tau/2}^{t_0+\tau/2}$$

$$= \frac{A}{j2\pi f}\left(e^{j2\pi f\tau/2} - e^{-j2\pi f\tau/2}\right)e^{-j2\pi ft_0}$$

$$= \frac{A\tau\sin(\pi f\tau)}{\pi f\tau} e^{-j2\pi ft_0}$$

$$= A\tau\text{sinc}(f\tau)e^{-j2\pi ft_0} \qquad \left(\text{sinc}\,(x) \triangleq \frac{\sin(\pi x)}{\pi x}\right)$$

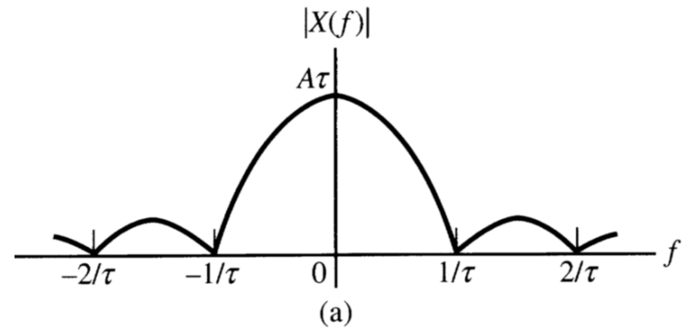
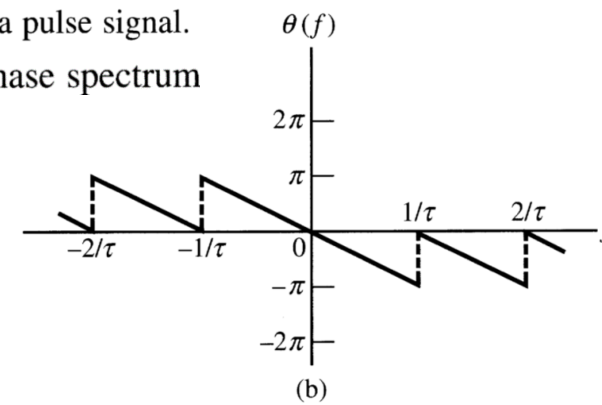


Figure 2.8
Amplitude and phase spectra for a pulse signal.
(a) Amplitude spectrum. (b) Phase spectrum
($t_0 = \frac{1}{2}\tau$ is assumed).



Fourier Transform of Singular Functions

$\delta(t)$ is not an energy signal, hence doesn't satisfy Dirichlet (pronounced dir ric clay) conditions, i.e. signal must be bounded, must be absolutely integrable, ...

However, its FT can be obtained by formal definition

$$\delta(t) \xrightarrow{FT} 1, \quad 1 \xrightarrow{FT} \delta(t)$$

$$A\delta(t-t_0) \xrightarrow{FT} Ae^{-j2\pi f t_0}, \quad Ae^{j2\pi f_0 t} \xrightarrow{FT} A\delta(f-f_0)$$

Example

Find CTFT of $\sum_n \delta(t - nT_0)$

$$\sum_n \delta(t - nT_0) = \sum_n X_n e^{j2\pi n f_0 t}$$

$$X_n = f_0 \int_{T_0} \delta(t) e^{-j2\pi n f_0 t} dt \overbrace{=}^{\text{sifting property}} f_0$$

$$x(t) = \sum_n X_n e^{j2\pi n f_0 t} = f_0 \sum_n e^{j2\pi n f_0 t}$$

$$F\left\{ f_0 \sum_n e^{j2\pi n f_0 t} \right\} = f_0 F\left\{ \sum_n e^{j2\pi n f_0 t} \right\}$$

$$= f_0 \sum_n \delta(f - n f_0)$$

Fourier Transform of Periodic Signals (Signals that are not energy signals)

Given a periodic $x(t)=\sum_n X_n e^{jn\omega_s t} \quad \leftrightarrow \quad X(f)=\sum_n X_n \delta(f-nf_s)$

Why?

Define $x(t)=\sum_n p(t-nT_s),\ \forall n,$ i.e. $p(t)\ =\ $ one period of $x(t)$

$$x(t)=\left[\sum_n \delta(t-nT_s)\right] * p(t)=\sum_n p(t-nT_s)$$

From convolution theorem

$$X(f)=F\left\{\sum_n \delta(t-nT_s)\right\}P(f)$$

$$=f_s\sum_n \delta(f-nf_s)P(f)$$

$$=f_s\sum_n P(nf_s)\delta(f-nf_s)$$

Poisson Sum Formula

Taking the inverse CTFT of $X(f)$

$$F^{-1}\{X(f)\} = x(t) = \sum_n p(t - nT_s)$$

$$= F^{-1}\left\{ f_s \sum_n P(nf_s)\delta(f - nf_s) \right\}$$

$$= \sum_n f_s P(nf_s) F^{-1}\left\{ \sum_n \delta(f - nf_s) \right\}$$

$$= \sum_n f_s P(nf_s) e^{j2\pi nf_s t}$$

Sampling of CTFT coeff. at nf_s

$$\Rightarrow \sum_n p(t - nT_s) = \sum_n f_s P(nf_s) e^{j2\pi nf_s t}$$ (Poisson sum formula)

Example

Given $x(t)=\sum_n \delta(t-nT_0)$. Find its CTFT.

Since $x(t)$ is periodic, it can be represented by its CTFS coefficients

$$x(t)=\sum_n \delta(t-nT_0)=\sum_n X_n e^{j2\pi n f_0 t}$$

$$X_n = f_0\int_{T_0}\delta(t)e^{-j2\pi n f_0 t}dt \overset{\text{sifting property}}{=} f_0$$

$$\Rightarrow x(t)=\sum_n X_n e^{j2\pi n f_0 t} = f_0\sum_n e^{j2\pi n f_0 t}$$

Using the Poisson sum formula

$$x(t)=\sum_n \delta(t-nT_0)=\sum_n f_0 P(nf_0)e^{j2\pi n f_0 t}$$

$P(nf_0)$ is the CTFT of $\delta(t)$, but sampled at nf_0.

From above, $\delta(t)\leftrightarrow 1$.

$$\Rightarrow \sum_n f_0 P(nf_0)e^{j2\pi n f_0 t} = \sum_n f_0 e^{j2\pi n f_0 t}$$

which agrees with the result of the CTFS.

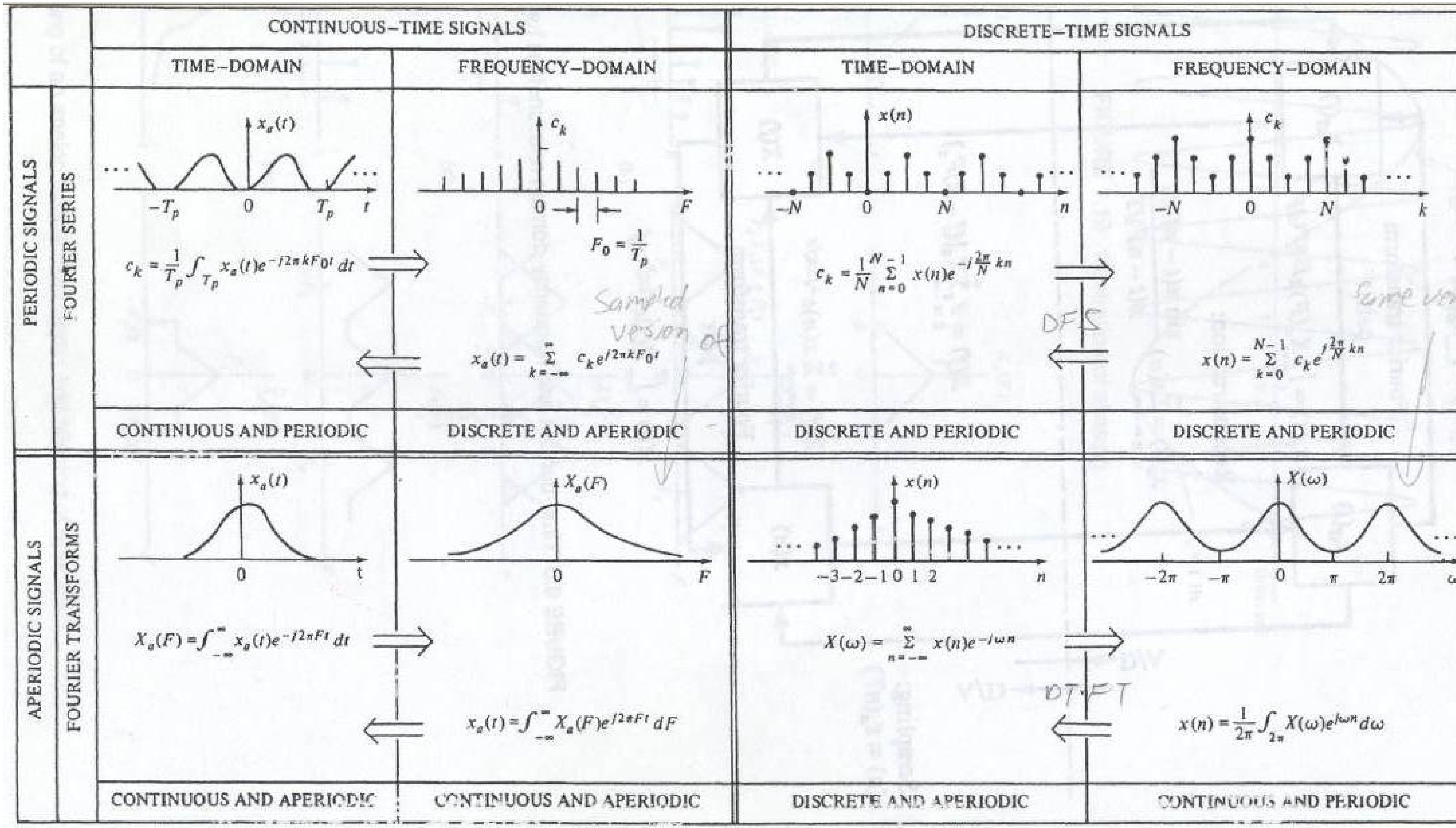


FIGURE 3.31 Summary of analysis and synthesis formulas.

Time-Average Correlation of Energy Signal (Deterministic Signals)

$$\phi(\tau) \triangleq F^{-1}\{G(f)\} = F^{-1}\{X(f)X^*(f)\} = F^{-1}\{X(f)\} * F^{-1}\{X^*(f)\}$$

$$= x(\tau) * x(-\tau) = \int_\lambda x(\lambda)x(\lambda+\tau)\,d\lambda = \lim_{T\to\infty}\int_{-T}^{T} x(\lambda)x(\lambda+\tau)\,d\lambda$$

Note:

Assuming $x(\tau)$ is real because

$$F\{x(-\tau)\} = \int_\tau x(-\tau)e^{-j2\pi f\tau}d\tau \overset{t=-\tau}{=} \int_t x(t)e^{j2\pi ft}d\tau \overset{x(\tau)\text{ real}}{=} \left(\int_t x(t)e^{-j2\pi ft}d\tau\right)^* = X^*(f)$$

Energy: $E = \phi(0) = \int_\lambda |x(\lambda)|^2\,d\lambda$

Time-Average Correlation of Power Signal (Deterministic Signals)

$$R(\tau) \triangleq \langle x(t), x(t+\tau) \rangle$$

$$= \begin{cases} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x^*(t+\tau)\,dt, & \text{for aperiodic signal} \\ \frac{1}{T_0}\int_{T_0} x(t)x^*(t+\tau)\,dt, & \text{for periodic signal} \end{cases}$$

$$R(0) = \langle x(t), x^*(t) \rangle = \left\langle |x(t)|^2 \right\rangle \triangleq \int_f S(f)\,df,$$

Power Spectral Density: $S(f) = F\{R(\tau)\}$ (Wiener-Khinchine theorem)

For periodic power signal

$$S(f) = F\{R(\tau)\} = \sum_n |X_n|^2 \delta(f - nf_0)$$

Interpretation of Time-Average Correlation

- $\phi(\tau)$ and $R(\tau)$ measure the similarity between the signal at time t and $t+\tau$
- $G(f)$ and $S(f)$ represent the signal energy or power per unit frequency at frequency f

Properties of Time-Average Correlation

- $R(0)$ = power = $<|x(t)|^2> \geq R(\tau), \forall \tau$, $\max\{R(\tau)\} = R(0)$
- $R(\tau)$ is even for real signals: $R(-\tau) = <x(t),x(t-\tau)> = R(\tau)$
- If $x(t)$ does not contain a period component, then

$$\lim_{|\tau|\to\infty} R(\tau) = \left\langle |x(t)|^2 \right\rangle$$

- If $x(t)$ is periodic with period T_0, then $R(\tau)$ is periodic in τ with the same period
- $S(f) = F\{R(\tau)\} \geq 0, \forall f$

Time-Average Crosscorrelation

Crosscorrelation of two power signals

$$R(\tau) \triangleq \langle x(t), y(t+\tau) \rangle = \langle x(t-\tau), y(t) \rangle$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, y^*(t+\tau)\, dt$$

Crosscorrelation of two energy signals

$$\phi_{xy}(\tau) = \int_t x(t)\, y^*(t+\tau)\, dt$$

Remarks:

$$R_{xy}(\tau) = R_{yx}^*(-\tau)$$

$$\phi_{xy}(\tau) = \phi_{yx}^*(-\tau)$$

I/O Relationships of LTI Systems for Time-Average Correlation

1. $R_{yx}(\tau) = h(\tau) * R_{xx}(\tau) = \int_{\lambda} h(\lambda) R_{xx}(\tau - \lambda) d\lambda$

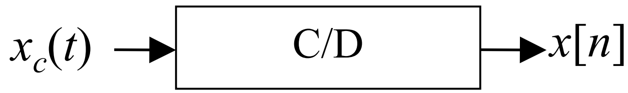
2. $R_{yy}(\tau) = h^{*}(-\tau) * h(\tau) * R_{xx}(\tau)$

3. $S_{yx}(f) = H(f) S_{xx}(f)$

4. $S_{yy}(f) = \left|H(f)\right|^{2} S_{xx}(f)$

Periodic Sampling

Ideal continuous-to-discrete-time (C/D) converter



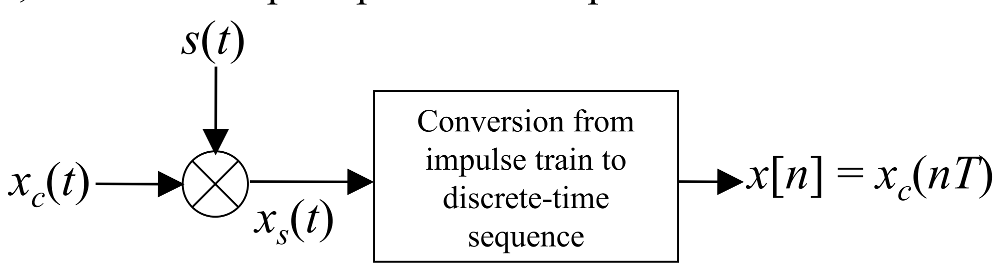
Continuous-time signal: $x_c(t)$

Discrete-time signal: $x[n] = x_c(nT)$, $-\infty < n < \infty$, T: sampling period

In theory, we break the C/D operation in two steps:

1. Ideal sampling using "analog delta function (Dirac delta function)"
   - Can be modeled by equations
2. Conversion from impulse train to discrete-time sequence
   - Only a concept, no mathematical model

In reality, the electronic analog-to-digital (A/D) circuits can approximate the ideal C/D operation. This circuitry is one piece; it cannot be split up into two steps



Note: DTFT

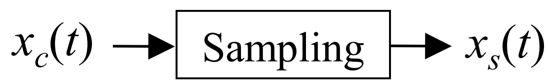
Analysis: $X\left(e^{j\omega}\right)=\sum_{n} x[n]e^{-j\omega n}$

Synthesis: $x[n]=\frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$

Signal analysis/projection and reconstruction concept identical to that of CTFT, but with some peculiarity concerning DT signals (more when you learn DSP)

We shall use $\Omega = 2\pi F$ to denote analog normalized frequency,
and
$\omega = 2\pi f$ to denote "digital" normalized frequency
(F = previous f, and f here is now "digital frequency")

Ideal Sampling – Time Domain



Ideal sampling signal: impulse train (continuous-time signal)

$$s(t) = \sum_n \delta(t - nT), \quad T\text{: sampling period}$$

Continuous - time signal: $x_c(t)$

Sampled (continuous - time) signal: $x_s(t)$

$$\begin{aligned} x_s(t) &= x_c(t)s(t) = x_c(t)\sum_n \delta(t - nT) \\ &= \sum_n x_c(t)\delta(t - nT) \\ &= \sum_n x_c(nT)\delta(t - nT) \end{aligned}$$

Ideal Sampling – Frequency Domain

Note that $$s(t) \leftrightarrow S(j\Omega) = \frac{2\pi}{T}\sum_k \delta(\Omega - k\Omega_s), \quad \text{where } \Omega_s = \frac{2\pi}{T}$$

Step 1: Ideal sampling (all in analog domain)

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{1}{T} X_c(j\Omega) * \sum_k \delta(\Omega - k\Omega_s)$$

$$= \frac{1}{T}\sum_k X_c(j\Omega) * \delta(\Omega - k\Omega_s)$$

$$\overbrace{=}^{\text{sifting property}} \frac{1}{T}\sum_k X_c\left(j(\Omega - k\Omega_s)\right) \qquad (*)$$

Remark: Ω: analog frequency (radians/sec)

ω: discrete (normalized) frequency (radians/sample)

$$\Omega = \frac{\omega}{T}; \quad -\pi < \omega \le \pi, \quad -\frac{\pi}{T} < \Omega < \frac{\pi}{T}$$

Step 1 (contd)

The sampled signal spectrum is the sum of shifted copies of the original.

Remark: In analog domain $x(t)\,y(t)$

$\Leftrightarrow \frac{1}{2\pi} X(j\Omega) * Y(j\Omega)$, $X_s(j\Omega)$ can also be expressed as:

$$X_s(j\Omega) = \int_t x_s(t) e^{-j\Omega t} dt = \int_t \sum_n x_c(nT)\delta(t - nT) e^{-j\Omega t} dt$$

$$= \sum_n x_c(nT) \int_t \delta(t - nT) e^{-j\Omega t} dt$$

$$= \sum_n x_c(nT) e^{-j\Omega nT} \qquad (**)$$

We also express $X(e^{j\omega})$ as:

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} = \sum_n x_c(nT) e^{-jwn} \qquad (***)$$

Step 2: Analog to Sequence (Analog to Discrete-Time)

Comparing (**) and (***), we see that $X\left(e^{j\omega}\right)$ is equivalent to $X_s\left(e^{j\omega}\right)$ if $\omega = \Omega T$, so that

$$X_s\left(j\Omega\right) = X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T} = X\left(e^{j\Omega T}\right)$$

Finally, from (*) we have

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_k X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

No mathematical model. The spectrum of $x_s\left(t\right)$, $X_s\left(j\Omega\right)$ has the same spectrum as $x\left[n\right]$ and $X\left(e^{j\Omega T}\right)$, respectively.

$X\left(e^{j\omega}\right)$ is a frequency-scaled version of $X\left(j\Omega\right)$

$$X\left(j\Omega\right) = X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T}.$$

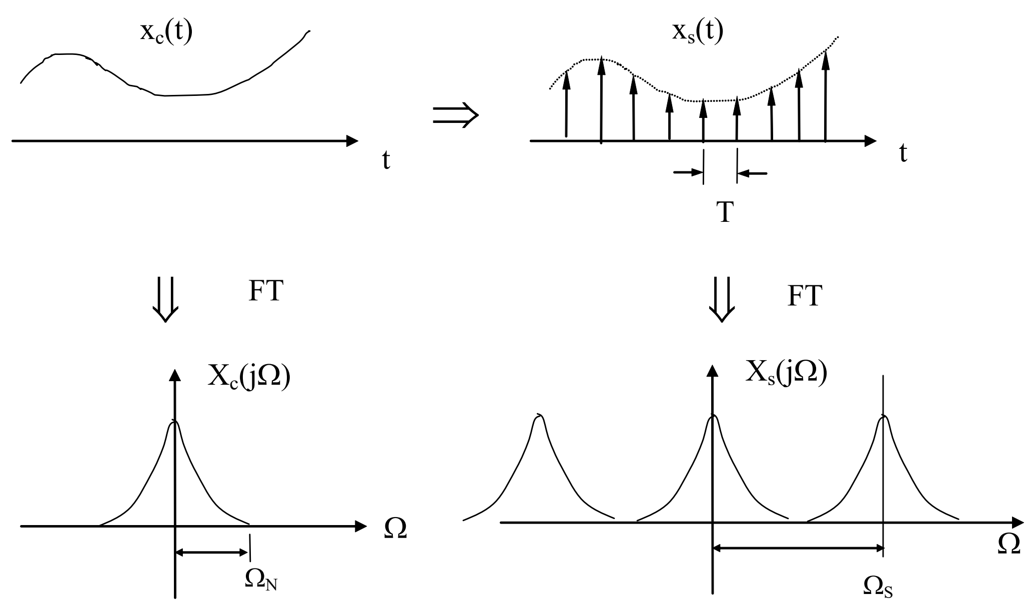
Since $X\left(e^{j\Omega T}\right) = \frac{1}{T}\sum_k X_c\left(j\left(\Omega - k\Omega_s\right)\right)$, thus

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_k X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

Remark: In time domain, $x_s\left(t\right)$ and $x\left[n\right]$ are two very different signals but have similar spectra in frequency domain.

Aliasing

- Two cases
  - No aliasing: $\Omega_s > 2\,\Omega_N$
  - Aliasing: $\Omega_s < 2\,\Omega_N$, where Ω_N is the highest nonzero frequency component of $X_c(j\Omega)$.
    - After sampling, the replicas of overlap (in frequency domain). That is, the higher frequency components of overlap with the lower frequency components of .

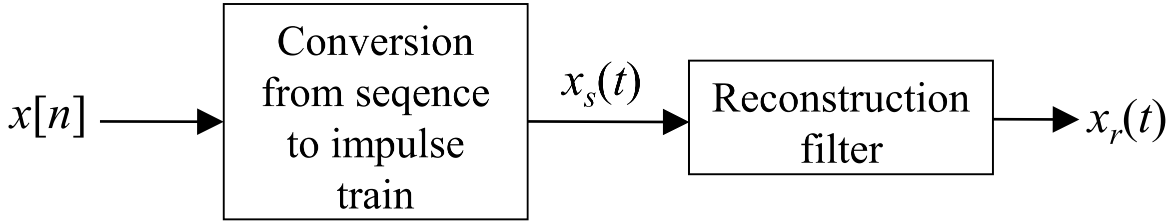


Nyquist Sampling Theorem

- Let $x(t)$ be a *bandlimited* signal with $X_c(j\Omega)=0$ for $|\Omega| \geq \Omega_N$. (i.e., no components at frequencies greater than Ω_N) Then $x_c(t)$ is uniquely determined by its samples $x[n]=x_c(nT)$, for $n=0, \pm 1, \pm 2, \ldots$, if $\Omega_s = 2\pi/T \geq 2\Omega_N$. (Nyquist, Shannon)
  - **Nyquist frequency** = Ω_N, the bandwidth of signal
  - **Nyquist rate** = $2\Omega_N$, the minimum sampling rate without distortion. (In some books, Nyquist frequency = Nyquist rate.)
    - Undersampling: $\Omega_s < 2\Omega_N$
    - Oversampling: $\Omega_s > 2\Omega_N$

Reconstruction of Bandlimited Signals

- Perfect reconstruction
  - Recovers the original continuous-time signal without distortion, e.g. ideal lowpass (bandpass) filter



- Based on frequency-domain analysis, if we can “clip” one copy of the original spectrum, $X_c(j\Omega)$, without distortion, we can achieve perfect reconstruction. For example, ideal lowpass filter, $h_r(t)$, can be used as a reconstruction filter
- Note that $x_s(t)$ is an analog signal

Signal Reconstruction Derivation

$$x_c(t) \rightarrow \text{sampling} \rightarrow x_s(t) = \sum_n x(nT)\delta(t-nT) \rightarrow \text{sequence conversion} \rightarrow x[n]$$

$$x[n] \rightarrow \text{impulse conversion} \rightarrow x_s(t) = \sum_n x[n]\delta(t-nT) \rightarrow \text{reconstruction} \rightarrow x_r(t)$$

$$x_r(t) = x_s(t) * h_r(t) = \int_\lambda \left\{ \sum_{n=-\infty}^{\infty} x[n]\delta(\lambda - nT) h_r(t-\lambda) \right\} d\lambda$$

$$= \sum_n \left\{ x[n] \int_\lambda \delta(\lambda - nT) h_r(t-\lambda) d\lambda \right\} = \sum_n x[n] h_r(t-nT)$$

Taking the Fourier transform of $x_r(t)$, we have

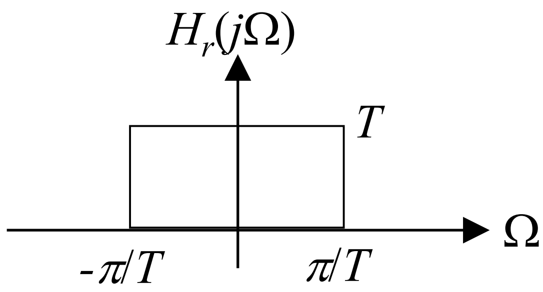
$$X_r(j\Omega) = \sum_n x[n] H_r(j\Omega) e^{-j\Omega T n} = H_r(j\Omega) \left\{ \sum_n x[n] e^{-j\Omega T n} \right\}$$

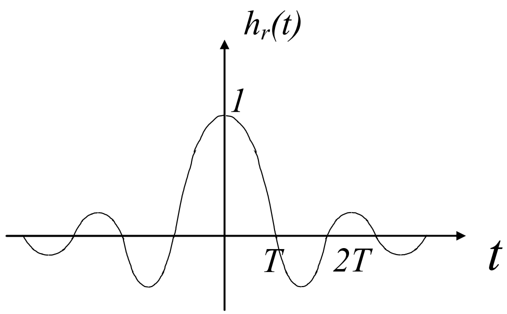
$$= H_r(j\Omega) X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T} = H_r(j\Omega) X\left(e^{j\Omega T}\right) = H_r(j\Omega) X(j\Omega)$$

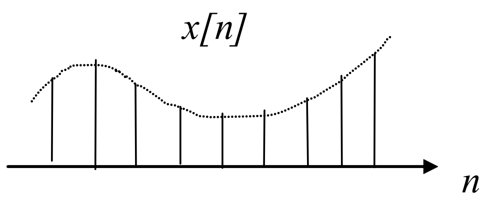
Ideal Lowpass Reconstruction Filter

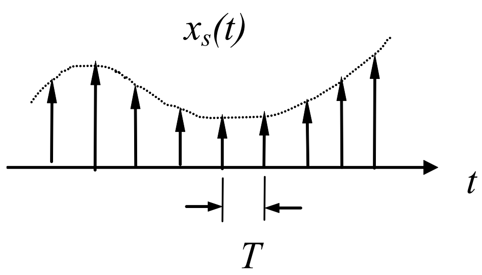
$$\text{Given: } H_r(j\Omega) = \begin{cases} T & -\pi/T < \Omega \le \pi/T \\ 0 & \textit{otherwise} \end{cases} \quad \Leftrightarrow \quad h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

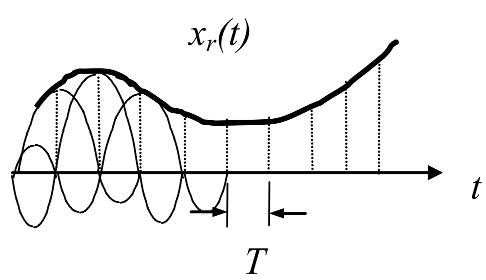
$$\text{Then: } x_r(t) = \sum_n x[n] \frac{\sin\left[\frac{\pi(t-nT)}{T}\right]}{\frac{\pi(t-nT)}{T}}$$


$H_r(j\Omega)$
T
Ω
-π/T
π/T


$h_r(t)$
1
T
2T
t


x[n]
n


$x_s(t)$
t
T


$x_r(t)$
t
T