Stochastic Processes

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Definition of Probability

- Random Processes (Stochastic Processes)
  - Informal definition
    - The outcomes (events) of a chance experiment are mapped into functions of time (waveforms)
    - Cf. Random variables: outcomes are mapped into numbers
  - Each waveform is called a sample function, or a realization. The totality of all sample functions is called an ensemble
  - Chance experiment that gives rise to this ensemble is called a random/stochastic process
  - Formal definition
    - Every outcome ζ we assign, according to a certain rule, a time function $X(t,\zeta)$. $X(t,\zeta_i)$ signifies a single time function
    - $X(t_j,\zeta)$ denotes a single RV
    - $X(t_j,\zeta_i)$ is a number

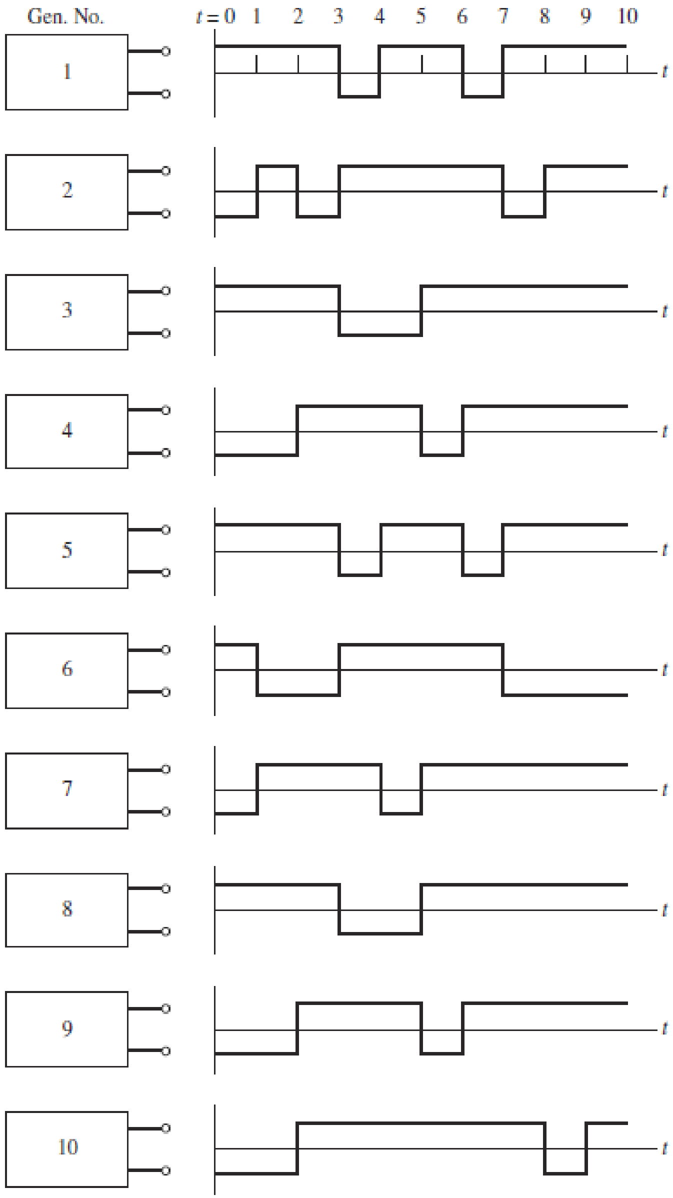


Figure 7.1 A statistically identical set of binary waveform generators with typical outputs.

Voltage at the terminals of a noise generator. 10 ensemble experiments

Statistical Description of Random Process

- A random process is statistically specified by its N^{th} order joint pdf's that describes a typical sample function at times $t_N > t_{N\text{-}1} > \ldots > t_1$, for any N where

$$F_{X1X2\ldots XN}(x_1,t_1;x_2,t_2;\ \ldots;\ x_N,t_N) = P(x_1\text{-}dx_1 < X_1 \leq x_1,\ x_2\text{-}dx_2 < X_2 \leq x_2,\ \ldots,\ x_N\text{-}dx_N < X_N \leq x_N)$$

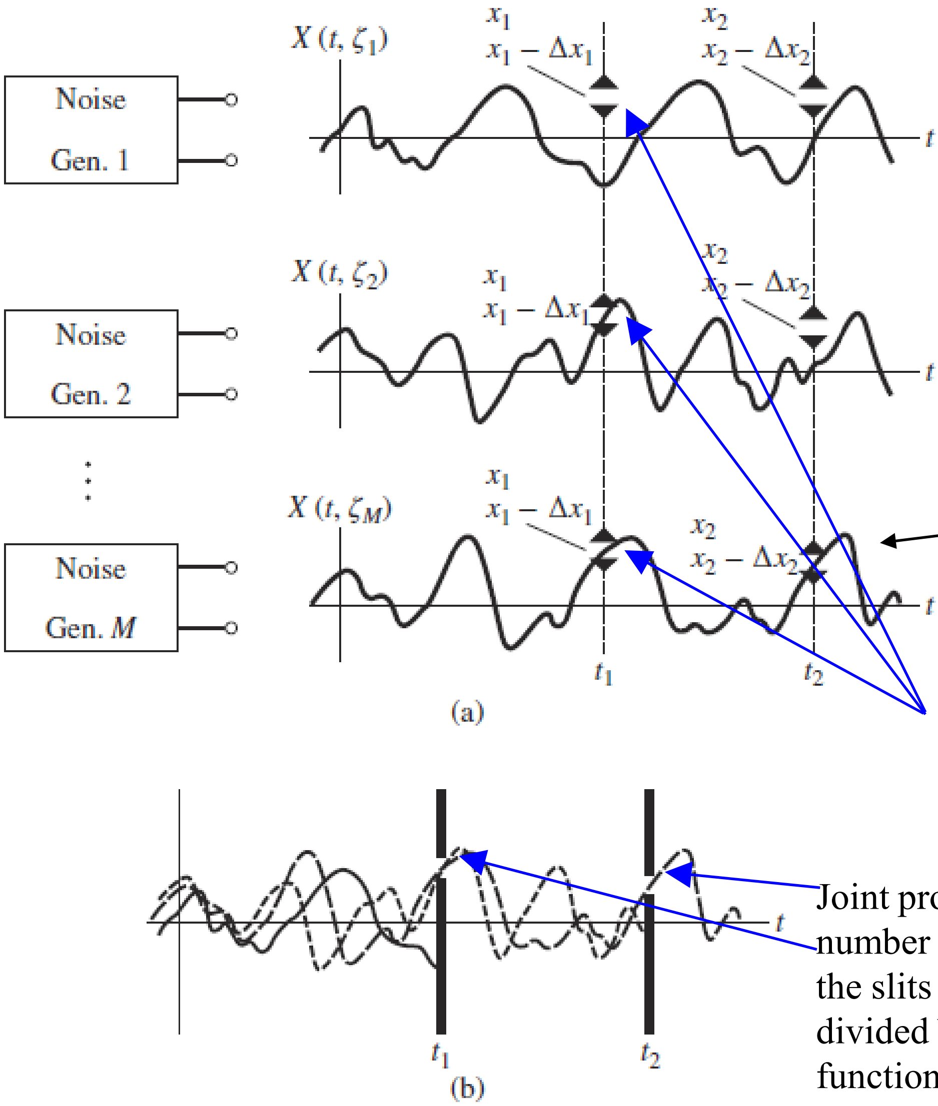


Figure 7.2
Typical sample functions of a random process and illustration of the relative-frequency interpretation of its joint pdf. (a) Ensemble of sample functions. (b) Superposition of the sample functions shown in (a).

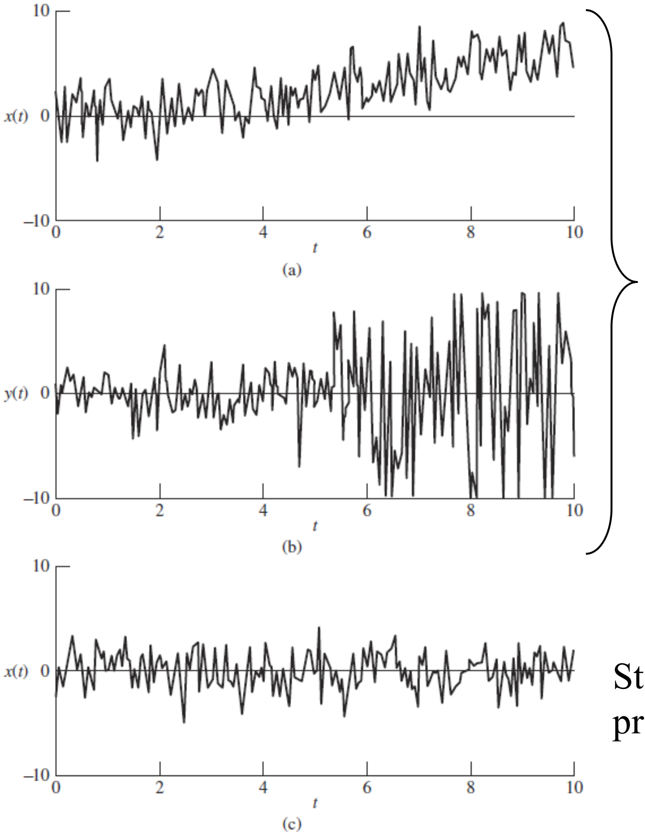
Random process from realization ζ_M.

$X(t_j,\zeta)$ is a random variable

Joint probability (from relative frequency) is the number of sample functions that pass through the slits placed at $t=t_1$ and $t=t_2$ in both barriers divided by the total number of M of sample functions as M becomes large w/o bound

Stationarity and Wide-Sense Stationarity

- Statistical stationarity in the strict sense or stationarity
  - Joint pdfs depend only on the time differences t_2-t_1, t_3-t_1, …, t_N-t_1
    - Not dependent on time origin
  - Mean and variance independent of time
  - Correlation coefficient or covariance depends only on difference, e.g. t_2-t_1
- Wide-sense stationarity (WSS)
  - Joint pdfs are dependent on time origin
  - Mean and variance independent of time
  - Correlation coefficient or covariance depends only on difference, e.g. t_2-t_1
- Stationarity ➔ WSS
  - Converse is not necessarily true
    - Exception: Gaussian random process (Why?)



Nonstationary processes

Stationary processes

Figure 7.3
Sample functions of nonstationary processes contrasted with a sample function of a stationary process. (a) Time-varying mean. (b) Time-varying variance. (c) Stationary.

Ensemble Average (Expectation)

Mean: $m_X(t) = E[X(t)] = \overline{X(t)} = \int_\alpha \alpha f_X(\alpha, t)\, d\alpha$

Variance: $\sigma_X^2(t) = E\left\{\left[X(t) - \overline{X(t)}\right]^2\right\} = E\left[|X(t)|^2\right] - \left|\overline{X(t)}\right|^2$

Covariance:

$$C_X(t_1, t_2) = E\left\{\left[X(t_1) - \overline{X(t_1)}\right]\left[X(t_2) - \overline{X(t_2)}\right]^*\right\}$$

$$= E\left[X(t_1)X^*(t_2)\right] - \overline{X(t_1)}\,\overline{X(t_2)}^*$$

$$C_X(t_2, t_1) = E\left\{\left[X(t_2) - \overline{X(t_2)}\right]\left[X(t_1) - \overline{X(t_1)}\right]^*\right\}$$

$$= E\left[X(t_2)X^*(t_1)\right] - \overline{X(t_2)}\,\overline{X(t_1)}^*$$

$\Rightarrow C_X(t_1, t_2) = C_X^*(t_2, t_1)$

Autocorrelation:

$$R_X(t_1, t_2) = E\left[X(t_1)X^*(t_2)\right]$$

$$= \int_{\alpha_2}\int_{\alpha_1} \alpha_1 \alpha_2 f_{X_1X_2}(\alpha_1, t_1; \alpha_2, t_2)\, d\alpha_1 d\alpha_2$$

Ensemble Average (Expectation) for WSS Process

WSS:

Mean: $m_X(t) = E\left[X(t)\right] = \text{constant}$

Variance: $\sigma_X^2(t) = \text{constant}$

Covariance:

$$C_X(\tau) \triangleq E\left\{\left[X(t) - \overline{X(t)}\right]\left[X(t+\tau) - \overline{X(t+\tau)}\right]^*\right\}$$

$$= E\left[X(t)X^*(t+\tau)\right] - \overline{X(t)}\,\overline{X(t+\tau)}^*$$

Autocorrelation:

$$R_X(\tau) \triangleq E\left[X(t)X^*(t+\tau)\right]$$

Ergodicity

Ergodic processes are processes for which time and ensemble averages are interchangeable.

For example, for real-valued WSS processes:

$$m_X = E\left[X(t)\right] = \left\langle X(t) \right\rangle$$

$$\sigma_X^2 = E\left\{\left[X(t) - \overline{X(t)}\right]^2\right\} = \left\langle \left[X(t) - \left\langle X(t) \right\rangle\right]^2 \right\rangle$$

$$R_X(\tau) = E\left[X(t)X(t+\tau)\right] = \left\langle X(t)X(t+\tau) \right\rangle,$$

where $\left\langle v(t) \right\rangle \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t)\,dt.$

Note:

- All time and ensemble averages are interchangeable, not just the above.
- Ergodicity $\Rightarrow$ strict-sense stationarity

Example 7.1

Consider a random process with sample function

$$n(t) = A\cos(2\pi f_0 t + \theta),$$

where f_0 is a constant and Θ is a RV with pdf

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & |\theta| \le \pi \\ 0, & \text{otherwise} \end{cases}.$$

Calculate its ensemble and time-average.

$$E[n(t)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos(2\pi f_0 t + \theta)\,d\theta = 0$$

$$\begin{aligned}\sigma_n^2(t) = E[n^2(t)] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [A\cos(2\pi f_0 t + \theta)]^2\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} A^2\cos^2(2\pi f_0 t + \theta)\,d\theta \\ &= \frac{A^2}{4\pi}\int_{-\pi}^{\pi} [1 + \cos(4\pi f_0 t + 2\theta)]\,d\theta \\ &= \frac{A^2}{2}\end{aligned}$$

$$\langle n(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(2\pi f_0 t + \theta)\,dt = 0$$

$$\begin{aligned}\langle n^2(t)\rangle &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A^2\cos^2(2\pi f_0 t + \theta)\,dt \\ &= \frac{A^2}{2}\end{aligned}$$

$E[n(t)] = \langle n(t)\rangle$=constant and $\sigma_n^2(t) = \langle n^2(t)\rangle$ = constant.

It may be stationary and ergodic.

Example 7.1

Suppose $f_\Theta(\theta) = \begin{cases} \frac{2}{\pi}, & |\theta| \le \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$.

Calculate its ensemble and time-average.

$$E\left[n(t)\right] = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} A\cos\left(2\pi f_0 t + \theta\right) d\theta$$

$$= \frac{2}{\pi} A\sin\left(2\pi f_0 t + \theta\right)\Big|_{-\pi/4}^{\pi/4} = \frac{2\sqrt{2}A}{\pi}\cos\left(2\pi f_0 t\right)$$

$$\sigma_n^2(t) = E\left[n^2(t)\right] = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\left[A\cos\left(2\pi f_0 t + \theta\right)\right]^2 d\theta$$

$$= \frac{A^2}{\pi}\int_{-\pi/4}^{\pi/4}\left[1 + \cos\left(4\pi f_0 t + 2\theta\right)\right] d\theta$$

$$= \frac{A^2}{2} + \frac{A^2}{\pi}\cos\left(4\pi f_0 t\right)$$

Process is not stationary as first and second moment depends on t, hence it is for different time origin.

Summary for Ergodic Process

1. Mean: $m_X(t) = E\left[X(t)\right] = \left\langle X(t) \right\rangle$ is the DC component

2. $\overline{X(t)}^2 = \left\langle X(t) \right\rangle^2$ is the DC power

3. $\overline{X^2(t)} = \left\langle X^2(t) \right\rangle$ is the total power

4. $\sigma_X^2(t) = \overline{X^2(t)} - \overline{X(t)}^2 = \left\langle X^2(t) \right\rangle - \left\langle X(t) \right\rangle^2$ is the power in the alternating current (time-varying) component

5. Total power $\overline{X^2(t)} = \sigma_X^2(t) + \left\langle X(t) \right\rangle^2$ is the AC power plus the DC power

Example 7.2: Random Telegraph (binary) Waveform

1. Values at any instant t_0 are either $X(t_0) = A$ or $X(t_0) = -A$ with equal probability

2. k number of switching instants in any time interval T obeys a Poisson distribution

$$P_T(k) = \frac{(\alpha T)^k}{k!} e^{-\alpha T}, \quad \text{for some } \alpha > 0$$

Probability of more than one switching instant at dt is zero. Probability of exactly one switching instant in dt is αdt, α is constant. Successive switchings are independent.

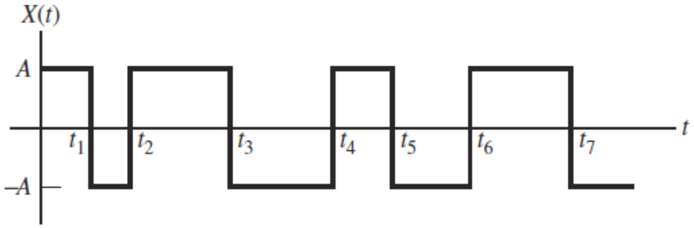


Figure 7.4
Sample function of a random telegraph waveform.

$$R_X(\tau) = E\left[X(t)X(t+\tau)\right]$$

$$= (A)(A) \bullet \Pr\left[X(t) \text{ and } X(t+\tau) \text{ have the same sign in interval } \tau\right]$$

$$+ (A)(-A) \bullet \Pr\left[X(t) \text{ and } X(t+\tau) \text{ have the different sign in interval } \tau\right]$$

$$= A^2 \bullet \Pr\left[\text{even \# of switching in interval } \tau\right] - A^2 \bullet \Pr\left[\text{odd \# of switching in interval } \tau\right]$$

Example 6.2: Random Telegraph (binary) Waveform

For $\alpha\tau > 0$

$$\Pr\left[\text{even \# of switching in interval } \tau\right] = \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{(\alpha\tau)^k}{k!} e^{-\alpha\tau}$$

$$= e^{-\alpha\tau} \sum_{k=0}^{\infty} \frac{1^k + (-1)^k}{2} \frac{(\alpha\tau)^k}{k!}$$

$$\left(\text{since } \sum_{k=0}^{\infty} \frac{(\alpha\tau)^k}{k!} = e^{\alpha\tau}\right) \qquad = \frac{e^{-\alpha\tau}}{2}\left(e^{\alpha\tau} + e^{-\alpha\tau}\right)$$

$$= \frac{1}{2}\left(1 + e^{-2\alpha\tau}\right)$$

Example 6.2: Random Telegraph (binary) Waveform

$$\Pr[\text{odd \# of switching in interval } \tau] = \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{(\alpha\tau)^k}{k!} e^{-\alpha\tau}$$

$$= e^{-\alpha\tau} \sum_{k=0}^{\infty} \frac{1^{k+1} + (-1)^{k+1}}{2} \frac{(\alpha\tau)^k}{k!} = e^{-\alpha\tau} \sum_{k=0}^{\infty} \frac{1^k - (-1)^k}{2} \frac{(\alpha\tau)^k}{k!}$$

$$= \frac{e^{-\alpha\tau}}{2}\left(e^{\alpha\tau} - e^{-\alpha\tau}\right)$$

$$= \frac{1}{2}\left(1 - e^{-2\alpha\tau}\right)$$

$$A^2 \bullet \Pr[\text{even \# of switching in interval } \tau] - A^2 \bullet \Pr[\text{odd \# of switching in interval } \tau]$$

$$= \frac{A^2}{2}\left(1 + e^{-2\alpha\tau}\right) - \frac{A^2}{2}\left(1 - e^{-2\alpha\tau}\right) = A^2 e^{-2\alpha\tau}$$

Similarly, for $\alpha\tau < 0$, $R_X(\tau) = A^2 e^{2\alpha\tau}$

$\therefore$ In general, $R_X(\tau) = A^2 e^{-2\alpha|\tau|}$

Correlation and Power Spectra

PSD: $S_X(f) = F\{R_X(\tau)\}$ for stationary process

Average Power: $R_X(0) = \int_f S_X(f)\,df$

What is the relationship between $S_X(f)$ and $F\{X(t)\}$?

Since sample functions of stationary random process are power signal, to consider its Fourier transform, let's define a truncated function

$$n_T(t,\zeta_i) = \begin{cases} n(t,\zeta_i), & |t| < \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases}.$$

$$\Leftrightarrow N_T(f,\zeta_i) = \int_{-T/2}^{T/2} n(t,\zeta_i) e^{-j2\pi ft} dt$$

So the time average power density over $\left[-\frac{T}{2},\frac{T}{2}\right]$ is $\frac{|N_T(f,\zeta_i)|^2}{T}$. For all ζ_i, take ensemble average and limit as $T \to \infty$ to obtain the distribution of power density with frequency, i.e.

$$S_n(f) = \lim_{T\to\infty} \frac{\overline{|N_T(f,\zeta_i)|^2}}{T}$$

Wiener-Khinchine Theorem

Show that $R_X(\tau) \Leftrightarrow S_X(f)$.

Rewriting the expression before: $S_n(f) = \lim_{T\to\infty} \dfrac{E\left[\left|F\{n_{2T}(t)\}\right|^2\right]}{2T}$,

$$\left|F\{n_{2T}(t)\}\right|^2 = \left|\int_{-T}^{T} n(t) e^{-j\omega t} dt\right|^2 = \int_{-T}^{T} n(t) e^{-j\omega t} dt \int_{-T}^{T} n^*(\sigma) e^{j\omega\sigma} d\sigma$$

$$\Rightarrow E\left[\left|F\{n_{2T}(t)\}\right|^2\right] = E\left[F\{n_{2T}(t)\} F^*\{n_{2T}(t)\}\right]$$

$$= \int_{-T}^{T}\int_{-T}^{T} E\left[n(t) n^*(\sigma)\right] e^{-j\omega(t-\sigma)} dt d\sigma$$

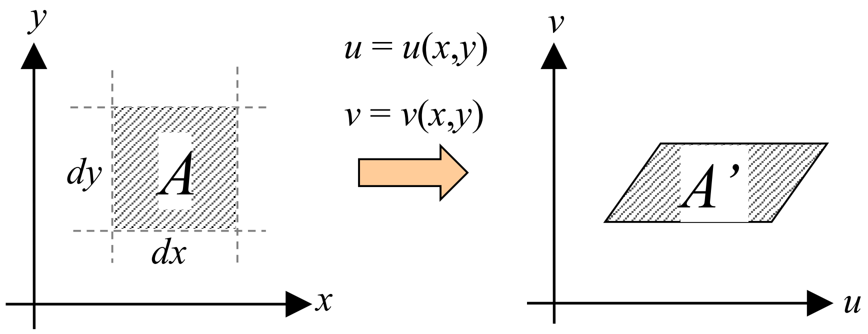
$$= \int_{-T}^{T}\int_{-T}^{T} R_n(t-\sigma) e^{-j\omega(t-\sigma)} dt d\sigma$$

Use variable substitution to solve.

Recall that an area $A = dt d\sigma \Leftrightarrow A' = du dv$, and that $du dv = \left|\det(\mathbf{J})\right| dx dy$,

where $\mathbf{J}$ is the Jacobian (similar to RV transformation in Ch. 5)

Digression: Review of Variable Substitution



In general, the area $A' = dudv = \left|\det\left(\mathbf{J}\right)\right| dxdy = \left|\det\left(\mathbf{J}\right)\right| A,$ where the $\left|\det\left(\mathbf{J}\right)\right|$ scales the original area A

Substitution using functions $u(x,y)$ and $v(x,y)$ which are linear equations w.r.t. x and y

In general, using linear approximation, for small Δx and Δy: $\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix},$

for small Δu and Δv, where $\mathbf{J} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{dx} & \frac{\partial u}{dy} \\ \frac{\partial v}{dx} & \frac{\partial v}{dy} \end{bmatrix}.$

$$dA' = \left|\det\left(\mathbf{J}\right)\right| dA = \left|\det\left(\mathbf{J}\right)\right| dxdy \quad \Rightarrow \int\int_A \cdots dxdy = \int\int_{A'} \cdots \frac{1}{\left|\det\left(\mathbf{J}\right)\right|} dudv$$

Wiener-Khinchine Theorem

$$\left|F\left\{n_{2T}(t)\right\}\right|^2=\left|\int_{-T}^{T}n(t)e^{-j\omega t}dt\right|^2=\int_{-T}^{T}n(t)e^{-j\omega t}dt\int_{-T}^{T}n^*(\sigma)e^{j\omega\sigma}d\sigma$$

$$\Rightarrow E\left[\left|F\left\{n_{2T}(t)\right\}\right|^2\right]=\int_{-T}^{T}\int_{-T}^{T}R_n(t-\sigma)e^{-j\omega(t-\sigma)}dtd\sigma$$

Let $u=t-\sigma,\ v=t \quad\Rightarrow\quad t=u+\sigma$ and $\sigma=t-u,$

$$\text{then } \left|\det(\mathbf{J})\right|=\left|\det\left(\begin{bmatrix}\frac{\partial u}{\partial t} & \frac{\partial u}{\partial\sigma}\\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial\sigma}\end{bmatrix}\right)\right|=\left|\det\left(\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\right)\right|=1$$

$$E\left[\left|F\left\{n_{2T}(t)\right\}\right|^2\right]=\int_{-T}^{T}\int_{-T}^{T}R_n(t-\sigma)e^{-j\omega(t-\sigma)}dtd\sigma$$

$$=\int\int_{A'}R_n(u)e^{-j\omega u}dvdu$$

lines where σ is constant

● u = -T+T = 0

● u = T+T = 2T

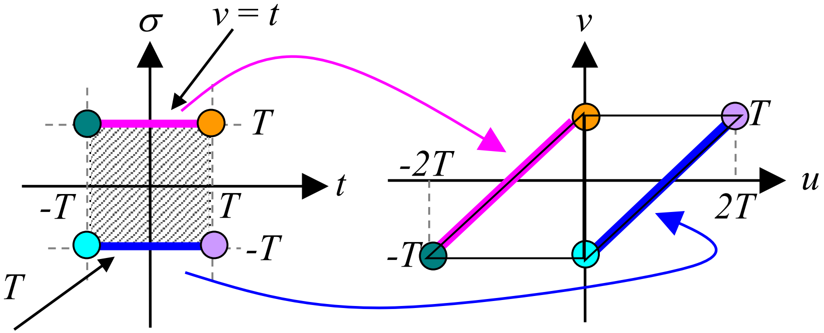
● u = -T-T = -2T

● u = T-T = 0

Lower and upper limits for u

u = t-T = v-T ➔ v = u+T

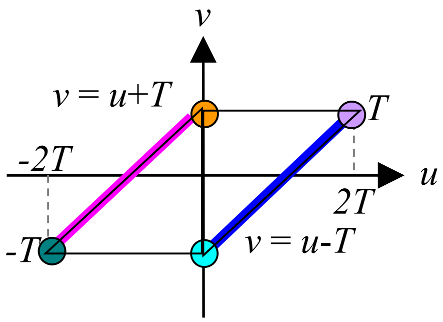
To obtain the limits of integration, consider the *uv* picture



u = t+T = v+T ➔ v = u-T

v = t

Wiener-Khinchine Theorem



$$E\left[\left|F\left\{n_{2T}(t)\right\}\right|^2\right] = \int_{-T}^{T}\int_{-T}^{T} R_n(t-\sigma)\, e^{-j\omega(t-\sigma)}\, dt\, d\sigma$$

$$= \int\int_{A'} R_n(u)\, e^{-j\omega u}\, dv\, du$$

$$= \int_{-2T}^{2T}\int_{u-T}^{u+T} R_n(u)\, e^{-j\omega u}\, dv\, du$$

$$= \int_{-2T}^{0}\int_{-T}^{u+T} R_n(u)\, e^{-j\omega u}\, dv\, du + \int_{0}^{2T}\int_{u-T}^{T} R_n(u)\, e^{-j\omega u}\, dv\, du$$

$$= \int_{-2T}^{0} R_n(u)\, e^{-j\omega u} \int_{-T}^{u+T} dv\, du + \int_{0}^{2T} R_n(u)\, e^{-j\omega u} \int_{u-T}^{T} dv\, du$$

$$= \int_{-2T}^{0} (2T+u)\, R_n(u)\, e^{-j\omega u}\, du + \int_{0}^{2T} (2T-u)\, R_n(u)\, e^{-j\omega u}\, du$$

$$= 2T\int_{-2T}^{2T} \left(1-\frac{|u|}{2T}\right) R_n(u)\, e^{-j\omega u}\, du$$

$$\Rightarrow S_n(f) = \lim_{T\to\infty} \frac{E\left[\left|F\left\{n_{2T}(t)\right\}\right|^2\right]}{2T} = \lim_{T\to\infty}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right) R_n(u)\, e^{-j\omega u}\, du = \int_u R_n(u)\, e^{-j\omega u}\, du$$

Example 7.4

Given the random process $n(t) = A\cos(2\pi f_0 t + \theta)$

where f_0 is a constant and Θ is a RV with pdf

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & |\theta| \le \pi \\ 0, & \text{otherwise} \end{cases}.$$

$$R_n(\tau) = E\left[n(t)n(t+\tau)\right] = \int_{-\pi}^{\pi} A^2 \cos(2\pi f_0 t + \theta)\cos\left[2\pi f_0(t+\tau) + \theta\right]\frac{d\theta}{2\pi}$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi} \cos(2\pi f_0 \tau) + \cos\left[2\pi f_0(2t+\tau) + 2\theta\right] d\theta$$

$$= \frac{1}{2}A^2\cos(2\pi f_0 \tau)$$

$$S_n(f) = F\left\{\frac{1}{2}A^2\cos(2\pi f_0 \tau)\right\} = \frac{A^2}{4}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

Properties of $R(\tau)$

(1) $R(0) \geq \left|R(\tau)\right|, \ \forall \tau$

Proof: Consider $\left|X(t) \pm X(t+\tau)\right|^2 \geq 0$ ($X(t)$ stationary)

$\Rightarrow \overline{X^2(t)} \pm 2\overline{X(t)X(t+\tau)} + \overline{X^2(t+\tau)} \geq 0$

$\Rightarrow 2R(0) \pm 2R(\tau) \geq 0$

$\Rightarrow -R(0) \leq R(\tau) \leq R(0)$

(2) $R(\tau)$ is even; $R(\tau) = R(-\tau)$ if $x(t)$ real

Proof: By definition (for WSS)

$$R(\tau) = \overline{X(t)X(t+\tau)} = \overline{X(t'-\tau)X(t')} = \overline{X(t')X(t'-\tau)} \triangleq R(-\tau)$$

with $t' = t + \tau$

Properties of $R(\tau)$

(3) $\lim_{|\tau|\to\infty} R(\tau) = \overline{X(t)}^2$ if $\{X(t)\}$ does not contain a periodic component

Proof: $\lim_{|\tau|\to\infty} R(\tau) = \lim_{|\tau|\to\infty} \overline{X(t)X(t+\tau)} \approx \overline{X(t)}\ \ \overline{X(t+\tau)} = \overline{X(t)}^2$

2nd equality is true because interdependence between $X(t)$ and $X(t+\tau)$ becomes less as $|\tau| \to \infty$, and last equalty is due to stationarity of $\{X(t)\}$

(4) If $\{X(t)\}$ periodic, then $R(\tau)$ is also periodic with same period.

Proof: $R(\tau) \triangleq E\left[X(t)X(t+\tau)\right] = E\left[X(t)X(t+T_0+\tau)\right] = R(T_0+\tau)$

(5) $S(f) = F\{R(\tau)\} \geq 0,\ \forall f$

Proof: From Wiener-Khinchine Theorem:

$$S(f) = \lim_{T\to\infty} \frac{1}{2T} E\left[\left|F\{X_{2T}(t)\}\right|^2\right] \geq 0$$

Properties of $S(f)$

(1) $S(f)=F\{R(\tau)\}\geq 0,\ \forall f$

(2) $S(f)$ is real-valued

Proof: because $R(\tau)$ is conjugate symmetric

(3) If $X(t)$ is real, $S(f)$ is even

Proof: If $X(t)$ is real, so is $R(\tau)$. FT of real-valued function, is even

(4) $\int_{\tau} R(\tau)d\tau = S(0)$

"total power" = $\int_{f} S(f)df = R(0)$

Example 7.5 – White Noise

Processes for which

$$S(f)=\begin{cases}\frac{N_0}{2}, & |f|\le B\\ 0, & \text{otherwise}\end{cases}$$

where N_0 is constant, are commonly referred to as bandlimited white noise.

As $B\to\infty$, all freqs are present, we called this process white. N_0 is the single-sided power spectral density of the nonbandlimited process.

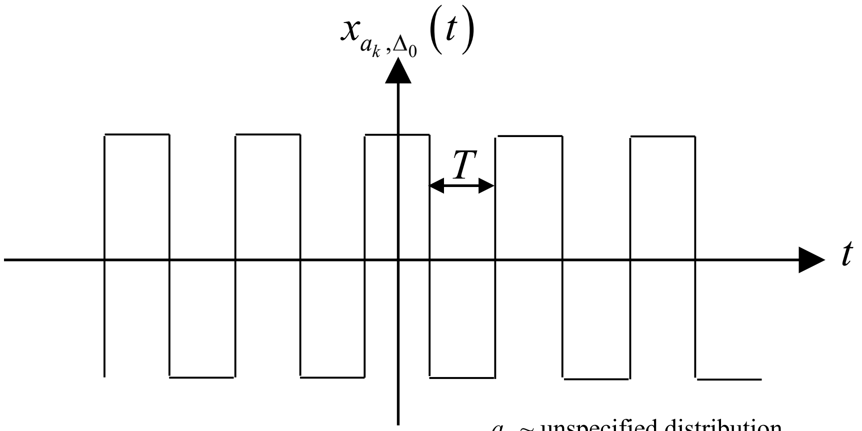
For a bandlimited process

$$\begin{aligned}R(\tau)&=\int_{-B}^{\mathrm{B}}\frac{N_0}{2}e^{j2\pi f\tau}df\\ &=\frac{N_0}{2}\frac{e^{j2\pi f\tau}}{j2\pi\tau}\Bigg|_{-B}^{B}=BN_0\frac{\sin(2\pi B\tau)}{2\pi B\tau}\\ &=BN_0\text{sinc}(2B\tau)\end{aligned}$$

As $B\to\infty$, $R(\tau)\to\frac{N_0}{2}\delta(\tau)$, i.e. samples are uncorrelated.

If Gaussian process, then samples are independent.

Autocorrelation Functions for Random Pulse Trains (Revisit)



Bandwidth requirement of line-coded data can be computed by looking at its PSD

$$x(t) \triangleq \sum_k a_k p(t - kT - \Delta)$$

Let $\ldots, a_{-1}, a_0, a_1, \ldots, a_k, \ldots$ be a sequence of RVs, indep with Δ, with correlation

$$E[a_k a_{k+m}] = \int_a a_k a_{k+m} p_{A_k}(a_k)\, da_k = R_m,\ m = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow R_{xx}(\tau) \triangleq E[x(t)x(t+\tau)]$$

$$= E\left[\sum_k \sum_m a_k a_{k+m} p(t - kT - \Delta)\, p(t + \tau - (k+m)T - \Delta)\right]$$

Indep. assumption →
$$= \sum_k \sum_m E[a_k a_{k+m}]\, E\left[\begin{matrix} p(t - kT - \Delta) \\ p(t + \tau - (k+m)T - \Delta) \end{matrix}\right]$$

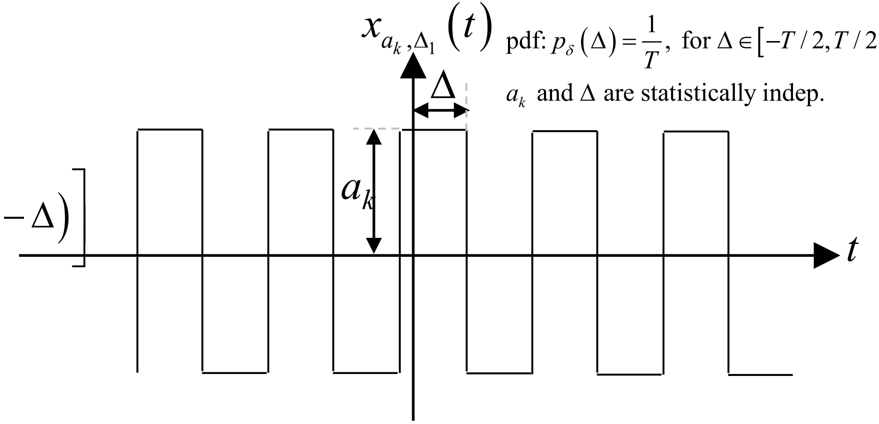
$$= \sum_m R_m \sum_k \frac{1}{T} \int_{\Delta=-T/2}^{T/2} p(t - kT - \Delta)\, p(t + \tau - (k+m)T - \Delta)\, d\Delta$$

$a_k \sim$ unspecified distribution

$\Delta \sim Unif[-T/2, T/2]$

pdf: $p_\delta(\Delta) = \frac{1}{T}$, for $\Delta \in [-T/2, T/2]$

a_k and Δ are statistically indep.



$$x_{a_k,\Delta_1}(t) \triangleq \sum_k a_k p(t - kT - \Delta_1)$$

Autocorrelation Functions for Random Pulse Trains (Revisit)

$$R_{xx}(\tau)=\sum_m R_m \sum_k \frac{1}{T}\int_{\Delta=-T/2}^{T/2} p(t-kT-\Delta)\,p(t+\tau-(k+m)T-\Delta)\,d\Delta$$

Let $u=t-kT-\Delta$

$$\Rightarrow R_{xx}(\tau)=\sum_m R_m \sum_k \frac{1}{T}\int_{u=t-(k-1/2)T}^{t-(k+1/2)T} p(u)\,p(u+\tau-mT)\,du$$

$$=\sum_m R_m\left[\frac{1}{T}\int_u p(u)\,p(u+\tau-mT)\,du\right]$$

$$=\sum_m R_m r(\tau-mT),$$

where $r(\tau)\triangleq\frac{1}{T}\int_u p(t)\,p(t+\tau)\,dt=\frac{1}{T}p(t)*p(-t)$

Example 7.6 – $\{a_k\}$ Has Memory

Suppose $\{a_k\}$ has memory built into it by the relationship

$$a_k = g_0 A_k + g_1 A_{k-1}$$

where g_0 and g_1 are constants, A_k are RVs, with $A_k = \pm A$. Sign is determined by a random coin toss independently from pulse to pulse for all k (note that if $g_1 = 0$, there is no memory). The assumed pulse shape is $p(t) = \prod(t/\tau)$.

Define $R_A(m) \triangleq E\left[A_k A_{k+m}\right] = \begin{cases} A^2, & m = 0 \\ 0, & m \neq 0 \end{cases}$

$$\begin{aligned} E\left[a_k a_{k+m}\right] &= E\left[\left(g_0 A_k + g_1 A_{k-1}\right)\left(g_0 A_{k+m} + g_1 A_{k+m-1}\right)\right] \\ &= E\left[g_0^2 A_k A_{k+m} + g_1^2 A_{k-1} A_{k+m-1} + g_0 g_1 A_k A_{k+m-1} + g_0 g_1 A_{k-1} A_{k+m}\right] \\ &= g_0^2 R_A(m) + g_1^2 R_A(m) + g_0 g_1 R_A(m-1) + g_0 g_1 R_A(m+1) \\ &= \begin{cases} \left(g_0^2 + g_1^2\right) A^2, & m = 0 \\ g_0 g_1 A^2, & m = \pm 1 \\ 0, & \text{otherwise} \end{cases} = R_m. \end{aligned}$$

Example 7.6 – $\{a_k\}$ Has Memory

Recall $r(\tau) \triangleq \frac{1}{T}\int_u p(t)\,p(t+\tau)\,dt = \frac{1}{T}p(t) * p(-t)$. Since $p(t) = \Pi\left(\frac{t}{T}\right)$

$$r(\tau) = \frac{1}{T}\int_t \Pi\left(\frac{t+\tau}{T}\right)\Pi\left(\frac{t}{T}\right)dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \Pi\left(\frac{t+\tau}{T}\right)dt = \Lambda\left(\frac{\tau}{T}\right)$$

Recall $R_{xx}(\tau) = \sum_m R_m r(\tau - mT)$,

$$\Rightarrow R_{xx}(\tau) = A^2\left\{\left(g_0^2 + g_1^2\right)\Lambda\left(\frac{\tau}{T}\right) + g_0 g_1\left[\Lambda\left(\frac{\tau+T}{T}\right) + \Lambda\left(\frac{\tau-T}{T}\right)\right]\right\}$$

$$\Rightarrow S_{xx}(f) = A^2 T\text{sinc}^2(fT)\left\{\left(g_0^2 + g_1^2\right) + g_0 g_1\left[e^{-j2\pi fT} + e^{j2\pi fT}\right]\right\}$$

$$= A^2 T\text{sinc}^2(fT)\left[g_0^2 + g_1^2 + 2g_0 g_1\cos(2\pi fT)\right]$$

Example 7.6 – $\{a_k\}$ Has Memory

$$S_{xx}(f) = A^2 T \text{sinc}^2(fT)\left[g_0^2 + g_1^2 + 2g_0 g_1 \cos(2\pi fT)\right]$$

Case 1: $g_0 = 1,\ g_1 = 0$ (no memory)

$$S_{xx}(f) = A^2 T \text{sinc}^2(fT)$$

Case 2: $g_0 = g_1 = 1/\sqrt{2}$

$$S_{xx}(f) = 2A^2 T \text{sinc}^2(fT)\cos^2(\pi fT)$$

Case 3: $g_0 = 1/\sqrt{2},\ g_1 = -1/\sqrt{2}$

In case 2, power spectrum is more confined.

Case 3: spectral width doubled, null at $f = 0$.

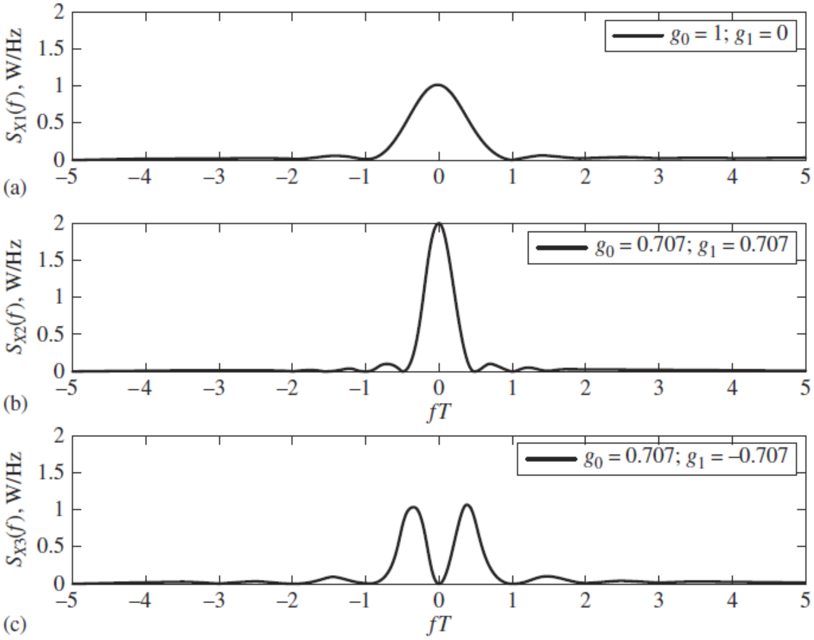


Figure 7.6
Power spectra of binary-valued waveforms. (a) Case in which there is no memory. (b) Case in which there is reinforcing memory between adjacent pulses. (c) Case where the memory between adjacent pulses is antipodal.

Cross-correlation

Given two random processes $X(t)$ and $Y(t)$, cross-correlation is defined as

$$R_{XY}(t_1,t_2) \triangleq E\left[X(t_1)Y^*(t_2)\right]$$

$$R_{XY}(\tau) \triangleq E\left[X(t)Y^*(t+\tau)\right] \text{ if } X(t) \text{ and } Y(t) \text{ are joint WSS}$$

Cross-covariance

$$C_{XY}(t_1,t_2) = E\left[\left(X(t_1)-m_X(t_1)\right)\left(X(t_2)-m_Y(t_2)\right)^*\right]$$
$$= R_{XY}(t_1,t_2) - m_X(t_1)m_Y^*(t_2)$$

If $X(t)$ and $Y(t)$ are joint WSS $S_{XY}(f) \triangleq F\left\{R_{XY}(\tau)\right\}$ is the cross-power spectral density

Properties of $R_{XY}(\tau)$ and $S_{XY}(f)$

(1) $R_{XY}(\tau) = R_{YX}^*(-\tau)$

(2) $S_{XY}(f) = S_{YX}^*(f)$

$S_{XY}(f) = S_{YX}(-f)$ if $X(t)$ and $Y(t)$ are real

Uncorrelated, Orthogonal, Independent Random Processes

Given two random processes $X(t)$ and $Y(t)$

(1) Uncorrelated

if $R_{XY}(t_1,t_2) = m_X(t_1) m_Y^*(t_1), \quad \forall t_1, t_2$

(2) Orthogonal

if $R_{XY}(t_1,t_2) = 0, \quad \forall t_1, t_2$

(3) Independence: if

$$f_{XY}(x_1, y_1, t_1; x_2, y_2, t_2; \ldots; x_n, y_n, t_n)$$
$$= f_X(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) f_Y(y_1, t_1; y_2, t_2; \ldots; y_n, t_n)$$

Remarks:

(1) Independence $\Rightarrow$ Uncorrelated

(2) Uncorrelated $\Rightarrow$ $\left(X(t) - m_X(t)\right)$ and $\left(Y(t) - m_Y(t)\right)$ are orthogonal

(3) (Uncorrelated and either $m_X(t) = 0$ or $m_Y(t) = 0$) $\Rightarrow$ orthogonal

(4) Uncorrelated and Gaussian $\Rightarrow$ Independent

Linear Systems and Random Processes

Given $h(t)$ is LTI, and $Y(t)=h(t)*X(t)$

Mean of $Y(t)$:

$$m_Y(t)=E\left[h(t)*X(t)\right]=E\left[\int_u h(u)X(t-u)du\right]=\int_u h(u)E\left[X(t-u)\right]du$$

$$=m_X(t)\int_u h(u)du=m_X(t)H(0)$$

Cross-correlation

$$R_{XY}(t_1,t_2)=E\left[X(t_1)Y(t_2)\right]=E\left[X(t_1)\int_u h(u)X(t_2-u)du\right]$$

$$=\int_u h(u)E\left[X(t_1)X(t_2-u)\right]du$$

$$=\int_u h(u)R_X(t_2-t_1-u)du$$

If $X(t)$ is WSS, let $\tau=t_2-t_1$

$$R_{XY}(\tau)=\int_u h(u)R_X(\tau-u)du=h(\tau)*R_X(\tau)$$

Linear Systems and Random Processes

Similarly

$$R_{YX}(t_1,t_2)=E\left[Y(t_1)X(t_2)\right]=E\left[\int_u h(u)X(t_1-u)du X(t_2)\right]$$

$$=\int_u h(u)E\left[X(t_1-u)X(t_2)\right]du$$

$$=\int_u h(u)R_X(t_2-t_1+u)du$$

If $X(t)$ is WSS, let $\tau=t_2-t_1$

$$R_{YX}(\tau)=\int_u h(u)R_X(\tau+u)du=h(-\tau)*R_X(\tau)=h(-\tau)*R_X(-\tau)=R_{XY}(-\tau)$$

$$R_Y(\tau)=E\left[Y(t)Y(t+\tau)\right]=E\left[\int_u h(u)X(t-u)du Y(t+\tau)\right]$$

$$=\int_u h(u)E\left[X(t-u)Y(t+\tau)\right]du$$

$$=\int_u h(u)R_{XY}(\tau+u)du$$

$$=h(-\tau)*R_{XY}(\tau)$$

$$=h(-\tau)*h(\tau)*R_X(\tau)$$

Linear Systems and Power Spectral Densities

$$R_{XY}(\tau)=h(\tau)*R_X(\tau) \Leftrightarrow S_{XY}(f)=H(f)S_X(f)$$

$$R_{YX}(\tau)=h(-\tau)*R_X(\tau)=R_{XY}(-\tau) \Leftrightarrow S_{YX}(f)=H(-f)S_X(f)=H^*(f)S_X(f)$$

$$R_Y(\tau)=h(-\tau)*h(\tau)*R_X(\tau) \Leftrightarrow S_Y(f)=H^*(f)H(f)S_X(f)=\left|H(f)\right|^2 S_X(f)$$

Remarks

- If $X(t)$ WSS, $h(t)$ LTI (no initial condition)
  - then $Y(t)$ is also WSS
- So far, we have only considered 2nd order statistics (mean, correlation, covariance). In general, given the joint pdf of $X(t)$, it is very difficult to find the joint pdf of $Y(t)$. But if $X(t)$ is jointly Gaussian, then $Y(t)$ is also jointly Gaussian and thus is completely characterized by mean and correlation functions

Filtered Gaussian Procesess

(1) Let $X(t)$ to be a stationary white Gaussian random process and supposed length of $h(t) > 1$.

Mean $= m_X$, autocorrelation $= R_X(\tau)$

$$R_X(t_1,t_2) = \delta(t_1 - t_2) = \delta(\tau)$$

$$S_X(f) = 1 \text{ (constant)}$$

$$\Rightarrow y(t) = \int_\tau x(\tau)h(t-\tau)d\tau$$

$$= \lim_{\Delta\tau \to 0} \sum_k x(k\Delta\tau)h(t - k\Delta\tau)\Delta\tau$$

$$= \text{weighted sum of Gaussian RVs}$$

$\therefore$ $Y(t)$ has a 1st order Gaussian distribution. Similarly the higher order joint pdf of $Y(t)$ is jointly Gaussian, but not white ($Y(t)$ has been colored by $h(t)$).

Filtered Gaussian Processess

(2) If input is not white, assuming the process is **regular** then it can be obtained by passing a white process through an **innovation** filter. Then by the same argument as in (1), $Y(t)$ is a colored Gaussian process.

What kind of processes are not regular processes?

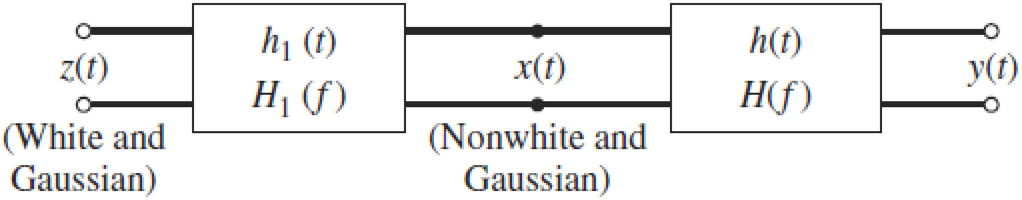


Figure 7.7
Cascade of two linear systems with Gaussian input.

$X(t)$ is a regular process

Properties of Gaussian Processes

(1) $X(t)$ Gaussian, $H(\bullet)$ stable and linear $\Rightarrow$ $Y(t)$ Gaussian

(2) $X(t)$ Gaussian and WSS $\Rightarrow$ $X(t)$ stationary in strict sense

(3) Samples of a Gaussian process $X(t_1), X(t_2), \ldots$ are uncorrelated

$\Rightarrow$ they are independent

(4) Samples of a Gaussian process, $X(t_1), X(t_2), \ldots$ have a joint Gaussian pdf specified completely by the set of means

$m_{X_i} = E\left[X(t_i)\right]$ and auto-covariance function

$$E\left[\left(X(t_i) - m_{X_i}\right)\left(X(t_i) - m_{X_i}\right)\right]$$

Remarks: Why do we use Gaussian model?

(1) Easy to analyze

(2) Central limit theorem: many "independent" events combined together become a Gaussian RV (random process)

Example 7.8 – RC Filter with WG Input

Let input to a lowpass RC filter be a zero mean white Gaussian process with PSD

$S_{n_i}(f) = \frac{N_0}{2}, -\infty < f < \infty.$ Output PSD is

$$S_{n_o}(f) = S_{n_i}(f)\left|H(f)\right|^2 = \frac{N_0/2}{1+(f/f_3)^2},$$

$f_3 = \frac{1}{2\pi RC}$ is the 3-dB cutoff freq.

$$R_{n_o}(\tau) = F^{-1}\left\{S_{n_o}(f)\right\} = \frac{\pi f_3 N_0}{2} e^{-2\pi f_3 |\tau|}$$

$$= \frac{N_0}{4RC} e^{-|\tau|/(RC)}, \quad \frac{1}{RC} = 2\pi f_3.$$

Output power: $\overline{n_o^2(t)} = \sigma_{n_o}^2 = R_{n_o}(0) = \frac{\pi f_3 N_0}{2} = \frac{N_0}{4RC}$

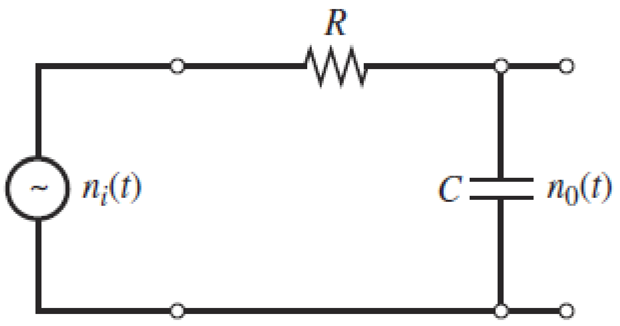


Figure 7.8

A lowpass RC filter with a white-noise input.

Example 7.8 – RC Filter with WG Input

Another approach:

$$\overline{n_o^2(t)} = \int_f S_{n_o}(f) e^{j2\pi f\tau} df \Big|_{\tau=0} = \int_f \frac{N_0/2}{1+(f/f_3)^2} df = \int_f \frac{N_0/2}{1+(2\pi RCf)^2} df$$

Let $x = 2\pi RCf$, $dx = 2\pi RC\ df$

$$\overline{n_o^2(t)} = \frac{N_0}{4\pi RC} \int_x \frac{1}{1+x^2} dx = \frac{N_0}{2\pi RC} \int_0^\infty \frac{1}{1+x^2} dx = \frac{N_0}{4RC}$$

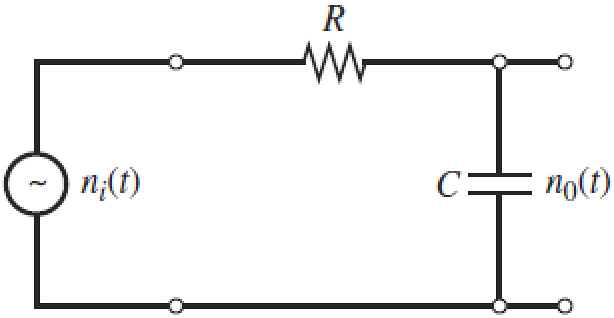


Figure 7.8
A lowpass RC filter with a white-noise input.

To obtain the PDF of $Y(t)$

Mean of $n_o(t)$: $\overline{n_o(t)} = 0 \cdot H(0) = 0$

Since $Y(t)$ Gaussian which has the form $\frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[\frac{-(x-m_x)^2}{2\sigma_x^2}\right]$

Then substituting the mean and variance from above we have

$$f_{n_o}(y,t) = f_{n_o}(y) = \frac{1}{\sqrt{\frac{\pi N_0}{2RC}}} \exp\left[\frac{-y^2}{\frac{N_0}{2RC}}\right]$$

Noise-Equivalent Bandwidth

- The noise-equivalent bandwidth for a lowpass filter is defined as the bandwidth of an ideal filter such that the power at the output of this filter, if excited by white Gaussian noise, is equal to that of the real filter given the same input signal
- The estimation of the noise-equivalent bandwidth allows us to compute the amount of in-band noise and its effect on the received signal SNR regardles of the filter's transfer function.

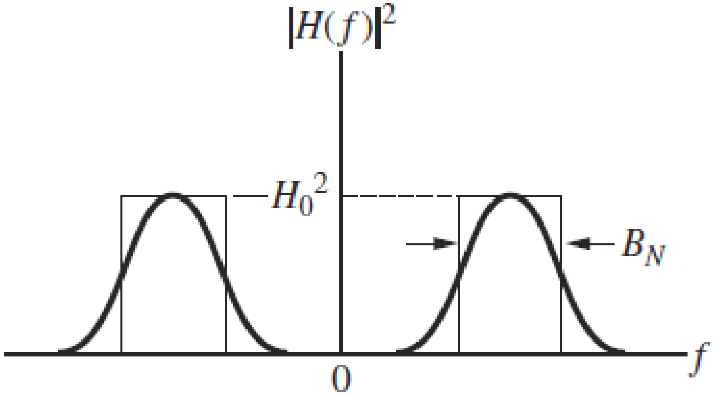


Figure 7.9
Comparison between $|H(f)|^2$ and an idealized approximation.

Noise-Equivalent Bandwidth

Suppose we pass white noise through a filter with frequency response $H(f)$

output average power:

$$P_{n_o} = R_{n_o}(0) = \int_f S_{n_o}(f)\,df = \int_f S_{n_i}(f)\left|H(f)\right|^2 df = \int_f \frac{N_0}{2}\left|H(f)\right|^2 df = N_0\int_0^{\infty}\left|H(f)\right|^2 df$$

$\frac{N_0}{2}$: 2-sided PSD of input

Suppose $H(f)$ is ideal with BW B_N and max gain H_0

$$\Rightarrow P_{n_o} = N_0\int_0^{\infty}\left|H(f)\right|^2 df = N_0\int_0^{B_N} H_0^2 df = N_0 B_N H_0^2$$

Question: What is the BW of an ideal, fictitious filter that has the same max. gain as $H(f)$ and that passes the same noise power?

Suppose the max. gain of $H(f)$ is H_0, the ans. is obtained by equating the results above.

Noise-Equivalent Bandwidth

Since $P_{n_o} = N_0 \int_0^\infty \left|H(f)\right|^2 df = N_0 \int_0^{B_N} H_0^2 df = N_0 B_N H_0^2$

$\Rightarrow B_N = \dfrac{1}{H_0^2} \int_0^\infty \left|H(f)\right|^2 df$ is the single-side BW of the fictitious filter

From Rayleigh's energy theorem, i.e. $\int_f \left|H(f)\right|^2 df = \int_t \left|h(t)\right|^2 dt$

and the fact that $H_0 = H(f)\Big|_{f=0} = \int_t h(t) e^{-j2\pi f t} dt \Big|_{f=0} = \int_t h(t) dt$

$$\Rightarrow B_N = \frac{1}{H_0^2} \int_t \left|h(t)\right|^2 dt = \frac{\frac{1}{2} \int_t \left|h(t)\right|^2 dt}{\left[\int_t h(t) dt\right]^2}$$

Example 7.10

The noise-equivalent BW of an n^{th} order Butterworth filter with squared mag. response

$$\left|H_n(f)\right|^2 = \frac{1}{1+\left(f/f_3\right)^{2n}}.$$

Note that $H_n(0)=1$

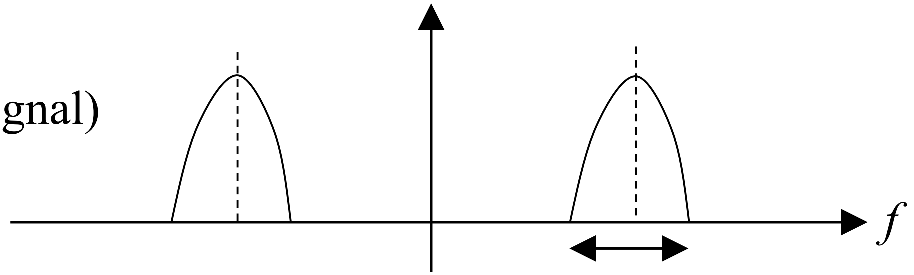
$$\Rightarrow B_N = \frac{1}{H_0^2}\int_0^\infty \left|H(f)\right|^2 df = \int_0^\infty \frac{1}{1+\left(f/f_3\right)^{2n}} df$$

$$= f_3\int_0^\infty \frac{1}{1+x^{2n}} dx$$

$$= \frac{\pi f_3/2n}{\sin\left(\pi/2n\right)}, \quad n=1,2,\ldots$$

Narrowband Noise

BP noise (signal)



$B << f_0$

B: BW of channel

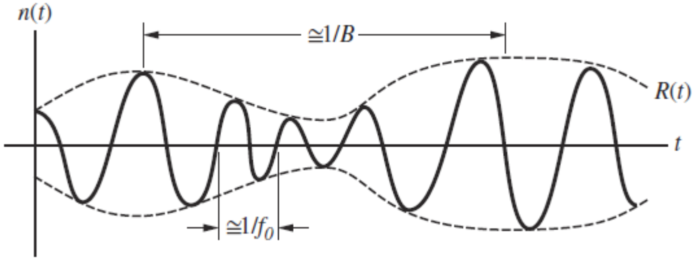
In communications, channel is characterized as a BP system, so it is more convenient to represent noise in terms of quadrature components

$$n(t) = n_c(t)\cos(2\pi f_0 t + \theta) - n_s(t)\sin(2\pi f_0 t + \theta)$$

$$R(t) = \sqrt{n_c^2(t) + n_s^2(t)}, \quad \phi(t) = \tan^{-1}\left(\frac{n_s(t)}{n_c(t)}\right)$$

θ: arbitrary time-invariant phase bias

Figure 7.11
A typical narrowband noise waveform.



Can also be represented using envelope-phase representation

$$n(t) = R(t)\cos(2\pi f_0 t + \theta)$$

For narrowband noise, $R(t)$ and $\phi(t)$ are slowly varying envelope and noise

Extract $n_c(t)$ and $n_s(t)$

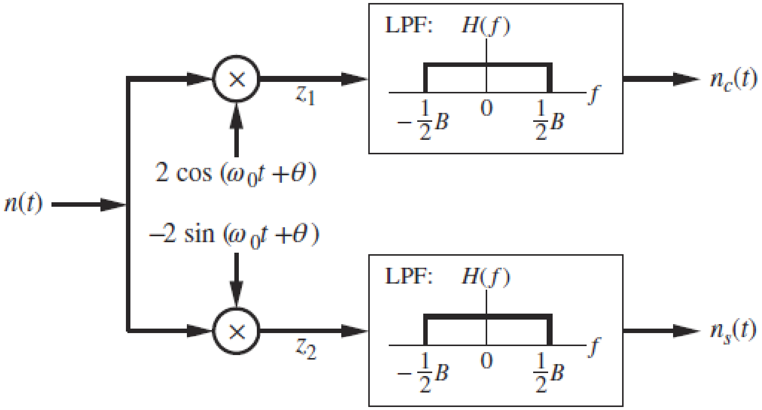


Figure 7.12
The operations involved in producing $n_c(t)$ and $n_s(t)$.

We will show that $n(t)=n_c(t)\cos(2\pi f_0 t+\theta)-n_s(t)\sin(2\pi f_0 t+\theta)$ in the mean-squared sense, i.e.

$$E\left[\left\{n(t)-\left[n_c(t)\cos(2\pi f_0 t+\theta)-n_s(t)\sin(2\pi f_0 t+\theta)\right]\right\}^2\right]=0$$

Remark: Here, we assume Θ is a RV, indep. of $n(t)$, uniformly distributed over $(0,2\pi)$ (or $(-\pi,\pi)$). If Θ is not a RV, $Z_1(t)$ and $Z_2(t)$ are not WSS, and LTI theory cannot be used to predict the outputs of LPFs.

Properties of Quadrature-component Representation

$$(1)\quad \overline{n(t)} = \overline{n_c(t)} = \overline{n_s(t)} = 0$$

Proof:

$$\text{(P1.1)}\quad \text{Let } X(t) \triangleq n(t) - \overline{n(t)}$$

$$R_X(\tau) = E\left[X(t)X(t+\tau)\right] = \overline{\left[n(t) - \overline{n(t)}\right]\left[n(t+\tau) - \overline{n(t+\tau)}\right]}$$

$$= \overline{n(t)n(t+\tau)} + \overline{n(t)} \bullet \overline{n(t+\tau)} - \overline{n(t)} \bullet \overline{n(t+\tau)} - \overline{n(t)} \bullet \overline{n(t+\tau)}$$

$$= R_n(\tau) - \left(\overline{n(t)}\right)^2 \quad \text{since } X(t) \text{ is WSS}$$

$$S_X(f=0) = \int_\tau R_X(\tau)\,d\tau = \int_\tau R_n(\tau) - \left(\overline{n(t)}\right)^2 d\tau$$

$$= S_n(0) - \int_\tau \left(\overline{n(t)}\right)^2 d\tau \overset{\text{BP noise}}{\overbrace{=}} 0 - \int_\tau \left(\overline{n(t)}\right)^2 d\tau \geq 0$$

The only possibility is $\left(\overline{n(t)}\right)^2 = 0$.

Properties of Quadrature-component Representation

$$E\left[\cos(2\pi f_0 t)\cos(\theta)-\sin(2\pi f_0 t)\sin(\theta)\right]=\frac{\cos(2\pi f_0 t)}{2\pi}\int_0^{2\pi}\cos(\theta)\,d\theta-\frac{\sin(2\pi f_0 t)}{2\pi}\int_0^{2\pi}\sin(\theta)\,d\theta$$

$$=\frac{\cos(2\pi f_0 t)}{2\pi}\left[\sin(\theta)\right]_0^{2\pi}+\frac{\sin(2\pi f_0 t)}{2\pi}\left[\cos(\theta)\right]_0^{2\pi}=0$$

(1) $\overline{n(t)}=\overline{n_c(t)}=\overline{n_s(t)}=0$

Proof:

(P1.2) $E\left[Z_1(t)\right]=2\overline{n(t)}\bullet\overline{\cos(2\pi f_0 t+\theta)}=0$

$$\Rightarrow E\left[n_c(t)\right]=E\left[Z_1(t)\right]H(0)=0$$

Similarly, $E\left[n_s(t)\right]=E\left[Z_2(t)\right]H(0)=0$

Properties of Quadrature-component Representation

(2) $S_{n_c}(f) = S_{n_s}(f) = LP\{S_n(f-f_0)+S_n(f+f_0)\}$

$$= \begin{cases} S_n(f-f_0)+S_n(f+f_0), & -W<f<W \\ 0, & \text{otherwise} \end{cases}$$

and $S_{n_c n_s}(f) = j \cdot LP\{S_n(f-f_0)-S_n(f+f_0)\}$

Proof: Show $S_{n_c}(f)$

(P2.1) Since $Z_1(t)=2n(t)\cos(\omega_0 t+\theta)$

$$R_{Z_1}(\tau) = E\{4n(t)n(t+\tau)\cos(\omega_0 t+\theta)\cos(\omega_0(t+\tau)+\theta)\}$$

$$= 2E\{n(t)n(t+\tau)\}\cos\omega_0\tau + 2E\{n(t)n(t+\tau)\cos(2\omega_0 t+\omega_0\tau+2\theta)\}$$

$$= 2R_n(\tau)\cos\omega_0\tau + 2E\{n(t)n(t+\tau)\}E\{\cos(2\omega_0 t+\omega_0\tau+2\theta)\}$$

$$= 2R_n(\tau)\cos\omega_0\tau$$

Thus,

$$S_{Z_1}(f) = S_n(f) * [\delta(f-f_0)+\delta(f+f_0)] = S_n(f-f_0)+S_n(f+f_0)$$

Properties of Quadrature-component Representation

(P2.2) $n_c(t)$ is the lowpass portion of $Z_1(t)$

$$\therefore\ S_{n_c}(f) = LP\left\{S_n(f - f_0) + S_n(f + f_0)\right\}$$

Similarly, $\therefore\ S_{n_s}(f) = LP\left\{S_n(f - f_0) + S_n(f + f_0)\right\}$

Show $S_{n_c n_s}(f)$

(P2.3) $R_{Z_1 Z_2}(\tau) = E\left[Z_1(t) Z_2(t+\tau)\right]$

$$= -E\left[\left(2n(t)\cos(\omega_0 t + \theta)\right)\left(2n(t+\tau)\sin(\omega_0 t + \omega_0 \tau + \theta)\right)\right]$$

$$= -2R_n(\tau)\sin(\omega_0 \tau)$$

$$\Rightarrow S_{Z_1 Z_2}(\tau) = j\left[S_n(f - f_0) - S_n(f + f_0)\right]$$

Properties of Quadrature-component Representation

$$
\begin{aligned}
\text{(P2.4)}\quad R_{n_c n_s}(\tau) &= E\left[n_c(t)n_s(t+\tau)\right] \\
&= E\left[\int_u h(u)Z_1(t-u)du\int_v h(v)Z_2(t+\tau-v)dv\right] \\
&= \int_v\int_u h(u)h(v)E\left[Z_1(t-u)Z_2(t+\tau-v)\right]dudv \\
&= \int_v\int_u h(u)h(v)R_{Z_1Z_2}(\tau+u-v)dudv \\
&= h(-\tau)*h(\tau)*R_{Z_1Z_2}(\tau)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow S_{n_c n_s}(f) &= H(f)H^*(f)S_{Z_1Z_2}(f) = \left|H(f)\right|^2 S_{Z_1Z_2}(f) \\
&= j\left|H(f)\right|^2\left[S_n(f-f_0)-S_n(f+f_0)\right] \\
&= jLP\left\{S_n(f-f_0)-S_n(f+f_0)\right\}
\end{aligned}
$$

Properties of Quadrature-component Representation

(2) $S_{n_c}(f) = S_{n_s}(f) = LP\{S_n(f-f_0)+S_n(f+f_0)\}$

(3) $\overline{n^2(t)} = \overline{n_c^2(t)} = \overline{n_s^2(t)} \triangleq N$

Proof: $\overline{n_c^2(t)} = \int_f S_{n_c}(f)\,df = \int_f S_{n_s}(f)\,df = \overline{n_s^2(t)}$ (from (2))

$$= \int_f S_n(f)\,df = \overline{n^2(t)}$$

(4) If $S_n(f)$ is symmetric w.r.t. f_0, then $n_c(t_1)$ and $n_s(t_2)$ are uncorrelated for all t_1 and t_2

Proof: $S_n(f)$ is symmetric w.r.t. f_0

$\Rightarrow LP\{S_n(f-f_0)-S_n(f+f_0)\}=0$

(2) $S_{n_c n_s}(f) = j\cdot LP\{S_n(f-f_0)-S_n(f+f_0)\}$

$\Rightarrow R_{n_c n_s}(\tau)=0, \ \forall\tau$ (from (2))

$\Rightarrow n_c(t)$ and $n_s(t+\tau)$ are uncorrelated.

Properties of Quadrature-component Representation

(5) If $n(t)$ is Gaussian, $n_c(t_1)$ and $n_s(t_2)$ are Gaussian

Proof: $n_c(t_1)$ and $n_s(t_2)$ are weighted linear combination of $n(t)$

(6) If $n_c(t_1)$ and $n_s(t_2)$ are Gaussian and uncorrelated, their joint pdf is

$$f(n_c,t;n_s,t+\tau)=\frac{1}{2\pi N}e^{-\frac{n_c^2+n_s^2}{2\pi N}}=\frac{1}{\sqrt{2\pi N}}e^{-\frac{n_c^2}{2\pi N}}\bullet\frac{1}{\sqrt{2\pi N}}e^{-\frac{n_s^2}{2\pi N}}=f(n_c,t)\bullet f(n_s,t),\quad\forall\tau$$

implying $n_c(t)$ and $n_s(t)$ are independent (which further validates the property of Gaussian that if two Gaussian RPs/Rvs are uncorrelated, it implies they are also independent)

In terms of polar coordinates, $R(t)$ and $\phi(t)$

$$f(r,\phi)=\frac{r}{2\pi N}e^{-\frac{r^2}{2N}},\quad\forall r>0,\ |\phi|\le\pi$$

Narrowband Noise Model

Theorem: Given a WSS bandpass random process $n(t)$ with BW $= B$, then $n(t)$ can be represented by $n_c(t)\cos(2\pi f_0 t+\theta)-n_s(t)\sin(2\pi f_0 t+\theta)$ in mean-squared sense. That is

$$E\left[\left\{n(t)-\left[n_c(t)\cos(2\pi f_0 t+\theta)-n_s(t)\sin(2\pi f_0 t+\theta)\right]\right\}^2\right]=0.$$

Note that θ is a RV, uniformly distributed over $(-\pi,\pi)$ and is indep. of $n(t)$.

Proof: Let $\hat{n}(t)=n_c(t)\cos(2\pi f_0 t+\theta)-n_s(t)\sin(2\pi f_0 t+\theta)$

Wish to show $E\left\{\left[n(t)-\hat{n}(t)\right]^2\right\}=0$

$$E\left\{\left[n(t)-\hat{n}(t)\right]^2\right\}=\overline{n^2(t)}-2\overline{n(t)\hat{n}(t)}+\overline{\hat{n}^2(t)}$$

$$\overline{\hat{n}^2(t)}=E\left\{\left[n_c(t)\cos(2\pi f_0 t+\theta)-n_s(t)\sin(2\pi f_0 t+\theta)\right]^2\right\}$$

← 3rd term

$$=\overline{n_c^2(t)}\bullet\overline{\cos^2(2\pi f_0 t+\theta)}+\overline{n_s^2(t)}\bullet\overline{\sin^2(2\pi f_0 t+\theta)}-2\overline{n_c(t)n_s(t)}\bullet\overline{\cos(2\pi f_0 t+\theta)\sin(2\pi f_0 t+\theta)}$$

$$=\frac{1}{2}\overline{n_c^2(t)}+\frac{1}{2}\overline{n_s^2(t)}=\overline{n^2(t)}=N$$

Narrowband Noise Model

2nd term

$$\overline{n(t)\hat{n}(t)} = E\left\{n(t)\left[n_c(t)\cos(2\pi f_0 t+\theta) - n_s(t)\sin(2\pi f_0 t+\theta)\right]\right\}$$

$$\because n_c(t) = h(t)*\left[2n(t)\cos(2\pi f_0 t+\theta)\right] = \int_u h(t-u)2n(u)\cos(2\pi f_0 u+\theta)du$$

$$n_s(t) = -\int_u h(t-v)2n(v)\sin(2\pi f_0 v+\theta)dv$$

$$\Rightarrow \overline{n(t)\hat{n}(t)} = E\left\{\int_u h(t-u)2n(t)n(u)\cos(2\pi f_0 u+\theta)\cos(2\pi f_0 t+\theta)du\right\}$$

$$+E\left\{\int_v h(t-v)2n(t)n(v)\sin(2\pi f_0 v+\theta)\sin(2\pi f_0 t+\theta)du\right\}$$

$$=E\left\{\int_u h(t-u)n(t)n(u)\left[\cos(2\pi f_0(u-t))+\cos(2\pi f_0(u+t)+2\theta)\right]du\right\}$$

$$+E\left\{\int_v h(t-v)n(t)n(v)\left[\cos(2\pi f_0(v-t))-\cos(2\pi f_0(v+t)+2\theta)\right]du\right\}$$

$$\overset{n(t),\ \theta\text{ indep}}{\overbrace{=}} \int_u h(t-u)R_n(u-t)\cos(2\pi f_0(u-t))du$$

$$\cancel{+\int_u h(t-u)R_n(u-t)E\left[\cos(2\pi f_0(u+t)+2\theta)\right]du}$$

$$+\int_v h(t-v)R_n(v-t)\cos(2\pi f_0(v-t))dv$$

$$\cancel{-\int_v h(t-v)R_n(v-t)E\left[\cos(2\pi f_0(v+t)+2\theta)\right]dv}$$

Narrowband Noise Model

$$\overline{n(t)\hat{n}(t)} = \int_u h(t-u)R_n(u-t)\cos\left(2\pi f_0(u-t)\right)du + \int_v h(t-v)R_n(v-t)\cos\left(2\pi f_0(v-t)\right)dv$$

$$= 2\int_u h(t-u)R_n(t-u)\cos\left(2\pi f_0(t-u)\right)du \qquad \left(\because R_n(t-u) = R_n(u-t), \cos(x) = \cos(-x)\right)$$

$$= 2\int_\lambda h(\lambda)R_n(\lambda)\cos(2\pi f_0\lambda)d\lambda$$

From Parseval's Theorem: $\int_t x(t)y(t)dt = \int_f X(f)Y^*(f)df$

and $h(\lambda)\cos(2\pi f_0\lambda) \Leftrightarrow \frac{1}{2}H(f-f_0)+\frac{1}{2}H(f+f_0)$ and $R_n(\lambda) \Leftrightarrow S_n(f)$

$$\Rightarrow \overline{n(t)\hat{n}(t)} = \int_f \left[H(f-f_0)+H(f+f_0)\right]S_n(f)df$$

Since $S_n(f)$ is nonzero only when $H(f-f_0)+H(f+f_0)=1$ because of narrowband assumption

$$\Rightarrow \overline{n(t)\hat{n}(t)} = \int_f S_n(f)df = \overline{n^2(t)}$$

$$\therefore E\left\{\left[n(t)-\hat{n}(t)\right]^2\right\} = \overline{n^2(t)} - 2\overline{n(t)\hat{n}(t)} + \overline{\hat{n}^2(t)} = \overline{n^2(t)} - 2\overline{n^2(t)} + \overline{n^2(t)} = 0$$

Example 7.12

Consider a BP random process with PSD shown in Figure 7.13(a).

(1) If $f_0 = 7$ Hz, $S_n(f)$ symmetric w.r.t. f_0

$\Rightarrow$ $n_c(t)$ and $n_s(t)$ are uncorrelated

$S_{Z_1}(f)$ or $S_{Z_2}(f)$ shown in Figure 7.13(b),

shaded region $= LP\{S_{Z_1}(f)\} = S_{n_c}(f)$ or $S_{n_s}(f)$

(2) If $f_0 = 5$ Hz, $S_n(f)$ not symmetric w.r.t. f_0

$\Rightarrow$ $n_c(t)$ and $n_s(t)$ are correlated

$$S_{n_c}(f) = S_{n_s}(f) = LP\{S_n(f - f_0) + S_n(f + f_0)\}$$

$=$ shaded region in Figure 7.13(c)

$$-jS_{n_c n_s}(f) = LP\{S_n(f - f_0) - S_n(f + f_0)\} =$$

$=$ shaded region of Figure 7.13(d)

From the figure, we see that

$$S_{n_c n_s}(f) = 2j\left\{-\Pi\left(\frac{1}{4}(f-3)\right) + \Pi\left(\frac{1}{4}(f+3)\right)\right\}$$

$$\Leftrightarrow R_{n_c n_s}(\tau) = 2j\left[-4\text{sinc}(4\tau)e^{j6\pi\tau} + 4\text{sinc}(4\tau)e^{-j6\pi\tau}\right]$$

$$= 16\text{sinc}(4\tau)\sin(6\pi\tau)$$

So $n_c(t)$ and $n_s(t)$ are indeed correlated.

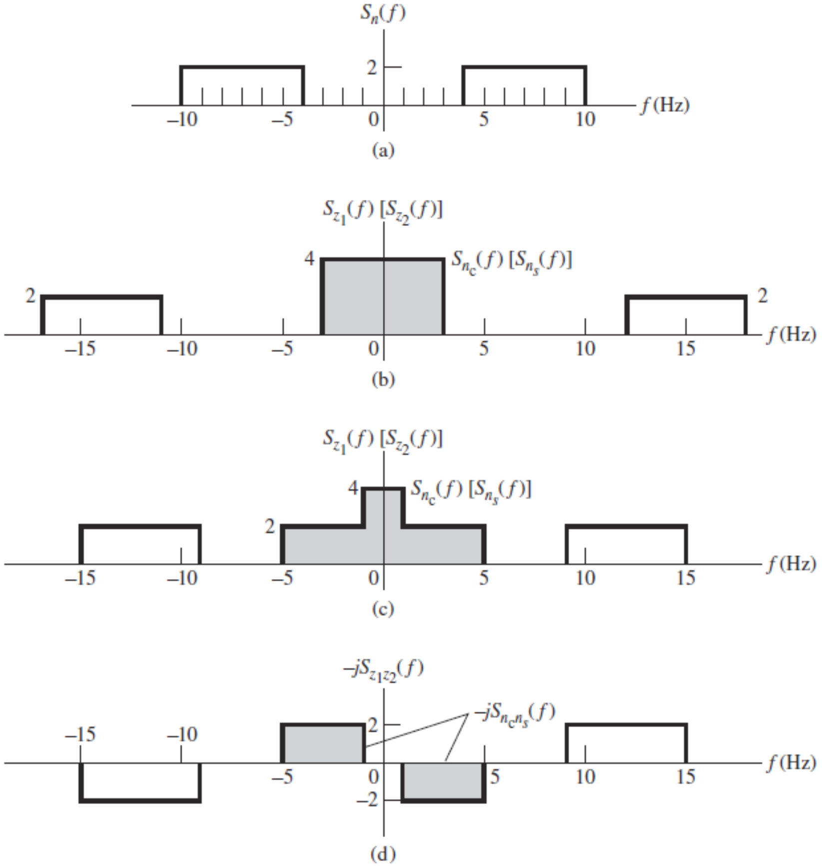


Figure 7.13
Spectra for Example 7.11. (a) Bandpass spectrum. (b) Lowpass spectra for $f_0 = 7$ Hz. (c) Lowpass spectra for $f_0 = 5$ Hz. (d) Cross spectra for $f_0 = 5$ Hz.