Noise in Modulation Systems

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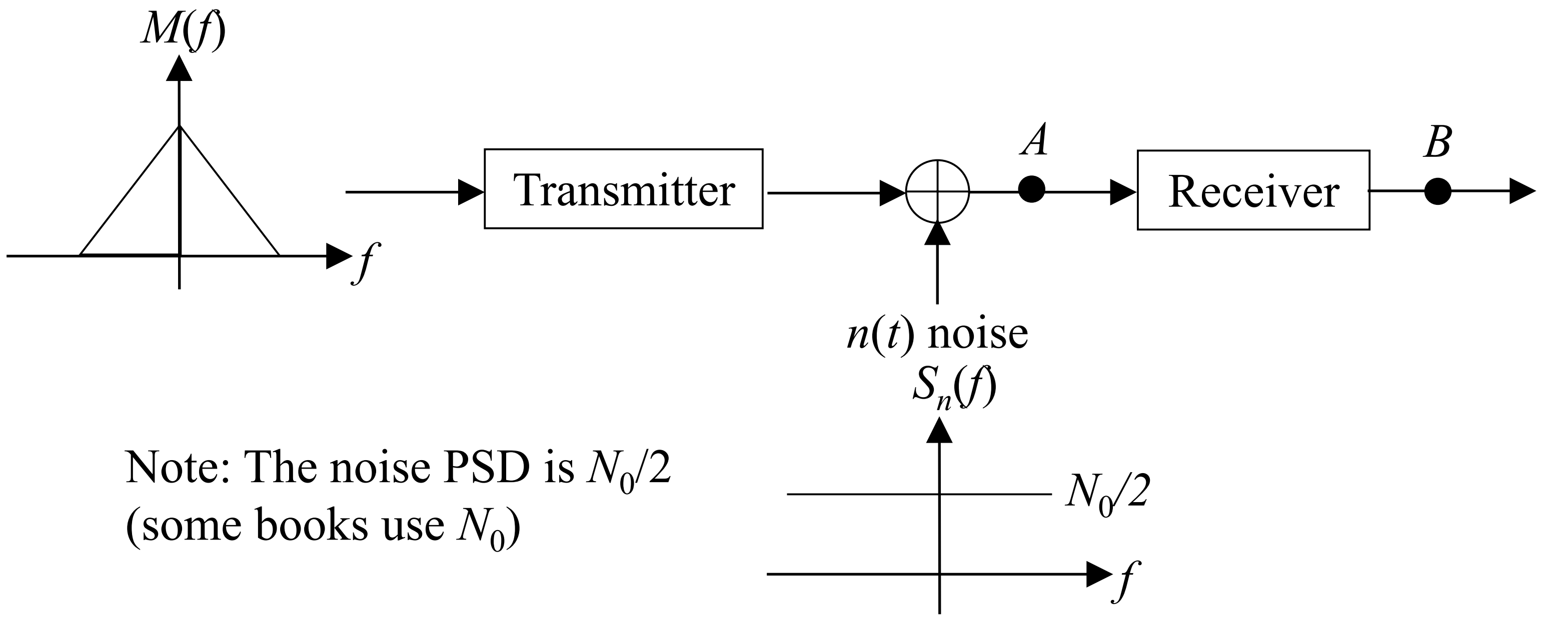
Signal-to-Noise Ratio (SNR)

- One of the most important design parameters in communication systems
  - Capacity depends on SNR
    - Scales logarithmically with SNR
  - Bit-error rate (BER) depends on SNR

$$SNR = \left.\frac{\text{Signal power}}{\text{Noise power}}\right|_{\text{at a certain point}}$$

- Other factors such as bandwidth (usually fixed by operator), diversity order, multiplexing gain are other important design parameters
- An interference limited system cannot only optimize SNR to sustain good performance

Channel Model



Note: The noise PSD is $N_0/2$
(some books use N_0)

N_0 is the average noise power per unit bandwidth (single-sided PSD) at A

Goal:

In general, design transceiver such that SNR at B is better than at A.

In this course, we focus only on the receiver

Baseband Model

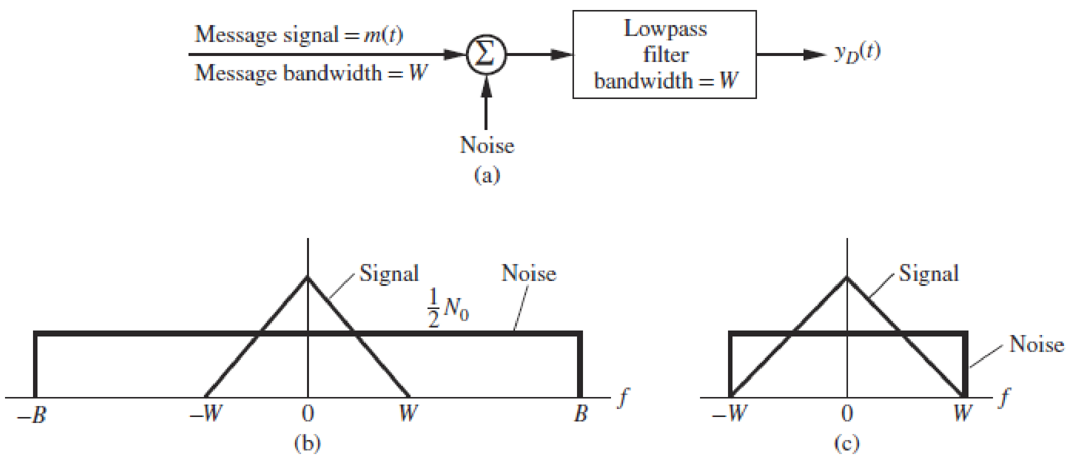


Figure 8.1
Baseband system. (a) Block diagram. (b) Spectra at filter input. (c) Spectra at filter output.

This is a basic assumption in many communication systems: We, at least, filter out the out-of-band noise. The filtering usually does not change signal power.

Message signal power = P_T (Watts) (transmit power)

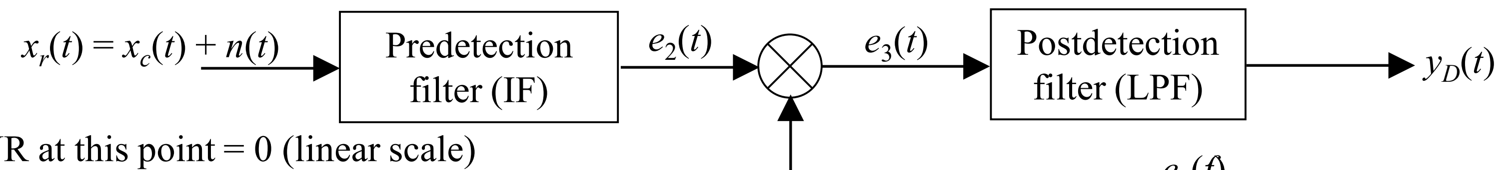
Assume ideal LPF with BW = W is used at Rx

Noise power at the filter input: $\int_{-B}^{B} \frac{N_0}{2} df = N_0 B$ $\quad (\text{SNR})_i = \frac{P_T}{N_0 B}$

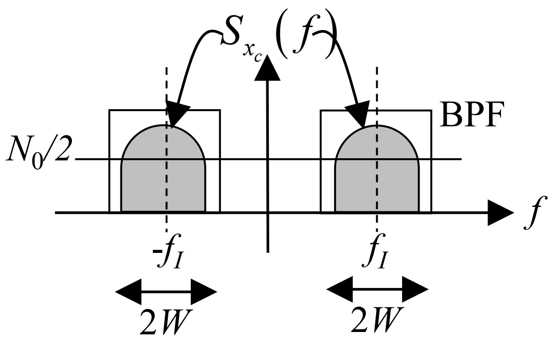
Noise power at the filter output: $\int_{-W}^{W} \frac{N_0}{2} df = N_0 W$ $\quad (\text{SNR})_o = \frac{P_T}{N_0 W}$

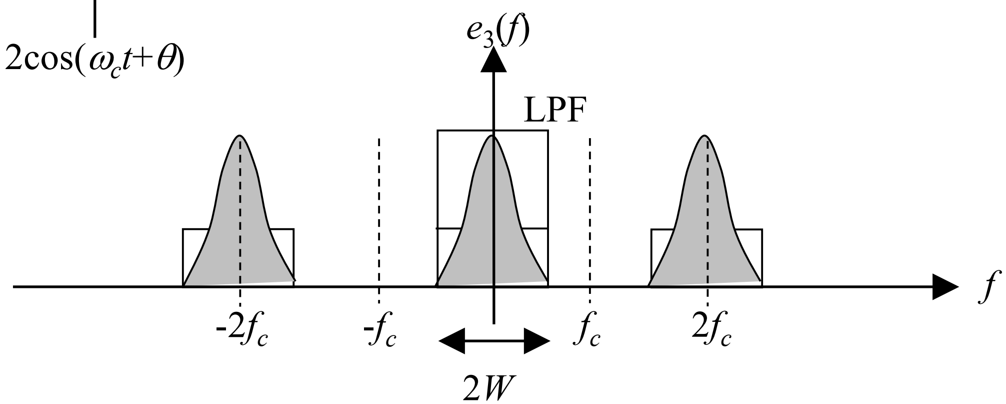
$\Rightarrow$ SNR enhancement $\frac{(\text{SNR})_o}{(\text{SNR})_i} = \frac{B}{W}$

Double-Sideband Systems – Coherent Detection



SNR at this point = 0 (linear scale)
because (white) noise's power is infinite





$$x_r(t) = A_c m(t)\cos(\omega_c t+\theta) + \boxed{n(t)}$$

white noise

BPF

Message passes through

BP noise

$$e_2(t) = A_c m(t)\cos(\omega_c t+\theta) + \boxed{n_c(t)\cos(\omega_c t+\theta) - n_s(t)\sin(\omega_c t+\theta)}$$

DSB – Coherent Detection

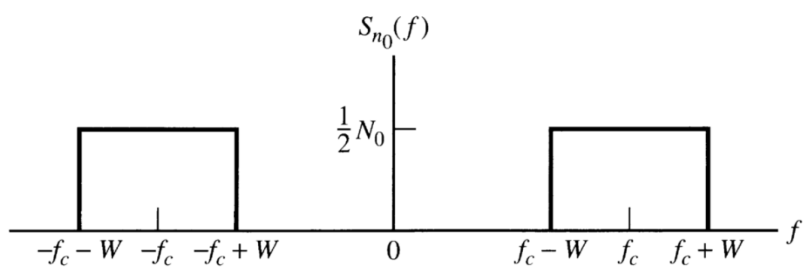
Noise power of $e_2(t)$:

$$\overline{n_0^2(t)} = BP\left\{E\left\{\left[n_c(t)\cos(\omega_c t+\theta) - n_s(t)\sin(\omega_c t+\theta)\right]^2\right\}\right\}$$

$$= BP\left\{\begin{array}{l} R_{n_c}(0)\cos^2(\omega_c t+\theta) \\ \cancel{-2R_{n_c n_s}(0)E\left[\cos(\omega_c t+\theta)\sin(\omega_c t+\theta)\right]} \\ +R_{n_s}(0)\sin^2(\omega_c t+\theta) \end{array}\right\}$$

$$= BP\left\{\begin{array}{l} \dfrac{R_{n_c}(0)}{2} + \dfrac{R_{n_c}(0)}{2}\cos(2\omega_c t+2\theta) \\ +\dfrac{R_{n_s}(0)}{2} + \dfrac{R_{n_s}(0)}{2}\sin(2\omega_c t+2\theta) \end{array}\right\}$$

$$= \frac{R_{n_c}(0)}{2} + \frac{R_{n_s}(0)}{2} = 2N_0 W$$



(a)

DSB – Coherent Detection

Signal power of $e_2(t)$:

$$\left\langle \left[A_c m(t) \cos(\omega_c t + \theta) \right]^2 \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[A_c m(t) \cos(\omega_c t + \theta) \right]^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{A_c^2 m^2(t)}{2} + \cancel{\frac{A_c^2 m^2(t)}{2} \cos(2\omega_c t + 2\theta)} \, dt$$

$$= \frac{A_c^2}{2} \overline{m^2},$$

$$\overline{m^2} \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} m^2(t) \, dt$$

$$\therefore \text{ Predetection SNR: } \left(\text{SNR}\right)_T = \frac{\frac{A_c^2}{2}\overline{m^2}}{2N_0 W} = \frac{A_c^2 \overline{m^2}}{4N_0 W}$$

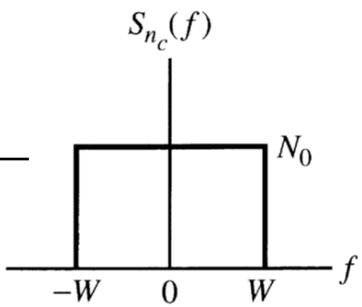
DSB – Coherent Detection

$$e_3(t) = e_2(t) 2\cos(\omega_c t + \theta)$$
$$= 2\cos(\omega_c t + \theta)\left[A_c m(t)\cos(\omega_c t + \theta) + n_c(t)\cos(\omega_c t + \theta) - n_s(t)\sin(\omega_c t + \theta)\right]$$
$$= A_c m(t)\left[1 + \cos(2\omega_c t + 2\theta)\right] + n_c(t)\left[1 + \cos(2\omega_c t + 2\theta)\right] - n_s(t)\sin(2\omega_c t + 2\theta)$$

After LPF:

$$y_D(t) = A_c m(t) + n_c(t)$$

Noise power of $y_D(t)$: $E\left[n_c^2(t)\right] = 2N_0 W$



(b)

Signal power of $y_D(t)$: $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[A_c m(t)\right]^2 dt = A_c^2 \overline{m^2}$

$$\overline{m^2} \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} m^2(t)\, dt \quad \text{(assume power signal)}$$

$\therefore$ Postdetection SNR: $(\text{SNR})_D = \dfrac{A_c^2 \overline{m^2}}{2N_0 W}$

$\therefore$ Detection gain $\triangleq \dfrac{(\text{SNR})_D}{(\text{SNR})_T} = \dfrac{A_c^2 \overline{m^2}}{2N_0 W}\dfrac{4N_0 W}{A_c^2 \overline{m^2}} = 2 = 3$ dB

DSB – Coherent Detection

Does this detection (demodulation) really provide a 3 dB gain? Not quite

(1) What is the "equivalent" baseband system?

Equating the baseband SNR $\gamma = \dfrac{P_T}{N_0 W}$ to $(\mathrm{SNR})_D = \dfrac{A_c^2 \overline{m^2}}{2N_0 W}$

we see that transmit power for the baseband system equals $\dfrac{1}{2} A_c^2 \overline{m^2}$

$\Rightarrow \gamma = (\mathrm{SNR})_D = \dfrac{A_c^2 \overline{m^2}}{2N_0 W}$ (baseband sys uses 1/2 Tx power and achieves same SNR)

(2) Why?

The baseband noise is $N_0 W$ (because the BW equals W).

However the passband noise is $2N_0 W$ (because its BW equals $2W$).

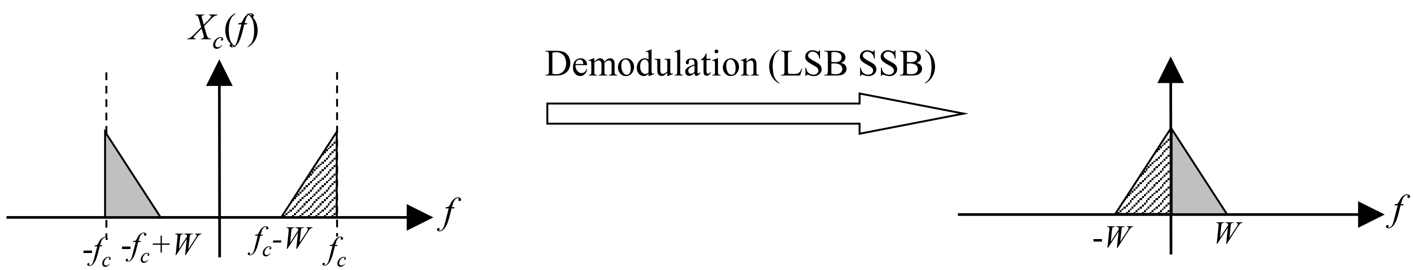
Thus, the detection gain (twice the Tx power cf. baseband sys) is just enough to cancel out the increase (compared to baseband noise) in passband noise due to its increased BW.

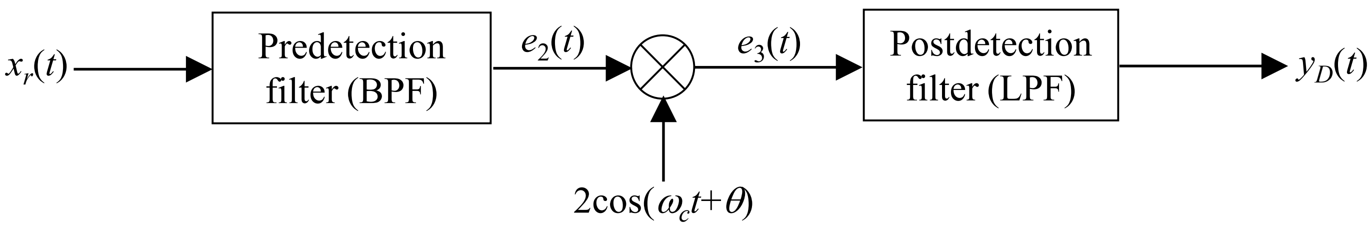
This analysis has assumed coherent detection, i.e. the LO signal at Rx $\cos(\omega_c t + \theta)$ has the same ω_c and θ as the carrier

Single-Sideband Modulation – Coherent Detection

$$x_r(t) = A_c\left[m(t)\cos(\omega_c t+\theta) \mp \hat{m}(t)\sin(\omega_c t+\theta)\right] + n(t),$$

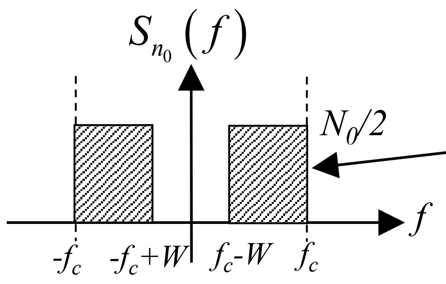
$+$: LSB SSB, $-$: USB SSB





SSB – Coherent Detection

Decompose noise into in-phase and quadrature components $\rightarrow$ BP noise



$$n(t) = n_c(t)\cos(\omega_c t+\theta) - n_s(t)\sin(\omega_c t+\theta)$$

$$\text{Let } N_T \triangleq \overline{n_0^2} = \overline{n_c^2} = \overline{n_s^2} = \frac{N_0}{2}2W = N_0 W$$

$\uparrow$ BP noise

Thus,

$$e_2(t) = A_c\left[m(t)\cos(\omega_c t+\theta) \mp \hat{m}(t)\sin(\omega_c t+\theta)\right]$$

$$+ n_c(t)\cos(\omega_c t+\theta) - n_s(t)\sin(\omega_c t+\theta)$$

$$= \left[A_c m(t) + n_c(t)\right]\cos(\omega_c t+\theta) \mp \underbrace{\left[A_c\hat{m}(t) - n_s(t)\right]\sin(\omega_c t+\theta)}_{\text{coherent demod removes quad comp.}}$$

$\downarrow$ mult. by $2\cos(\omega_c t+\theta)$ for demod

$e_3(t)$

$\downarrow$ LPF

$y_D(t) = A_c m(t) + n_c(t)$

SSB – Coherent Detection

$$\begin{cases} \text{Postdetection signal power: } s_D \triangleq A_c^2 \overline{m^2} \\ \text{Postdetection noise power: } N_D \triangleq \overline{n_c^2} = N_0 W \end{cases}$$

$$\text{Postdetection SNR: } (\text{SNR})_D = \frac{A_c^2 \overline{m^2}}{N_0 W}$$

What is the transmit power after predetection?

$$S_T = \overline{\left\{A_c\left[m(t)\cos(\omega_c t+\theta) \mp \hat{m}(t)\sin(\omega_c t+\theta)\right]\right\}^2}$$

$$= A_c^2\left\{\overline{\left[m(t)\cos(\omega_c t+\theta)\right]^2} + \overline{\left[\hat{m}(t)\sin(\omega_c t+\theta)\right]^2} - 2\overline{m(t)\hat{m}(t)\cos(\omega_c t+\theta)\sin(\omega_c t+\theta)}\right\}$$

$$\left(\text{Since } m(t) \text{ and } \hat{m}(t) \text{ are orthogonal, and } \cos(\omega_c t+\theta) \text{ and } \sin(\omega_c t+\theta) \text{ are orthogonal}\right)$$

$$= A_c^2\left\{\overline{m^2(t)}\,\overline{\cos^2(\omega_c t+\theta)} + \overline{\hat{m}^2(t)}\,\overline{\sin^2(\omega_c t+\theta)}\right\}$$

$$= A_c^2\left[\frac{\overline{m^2(t)}}{2} + \frac{\overline{\hat{m}^2(t)}}{2}\right] \qquad \left(\text{Since } \overline{m^2(t)} = \overline{\hat{m}^2(t)}\right)$$

$$= A_c^2 \overline{m^2(t)}$$

SSB – Coherent Detection

What is the noise power after predetection?

BW for pre- and postdetection equals W, so noise power is same as postdetection.

That is, $N_T = N_D = N_0 W$

$$\therefore \text{ Detection gain: } \frac{(\mathrm{SNR})_D}{(\mathrm{SNR})_T} = \frac{\dfrac{A_c^2 \overline{m^2}}{N_0 W}}{\dfrac{A_c^2 \overline{m^2}}{N_0 W}} = 1$$

What is the "equivalent" baseband system?

Equating baseband SNR $\gamma = \dfrac{P_T}{N_0 W}$ with $(\mathrm{SNR})_D = \dfrac{A_c^2 \overline{m^2}}{N_0 W}$

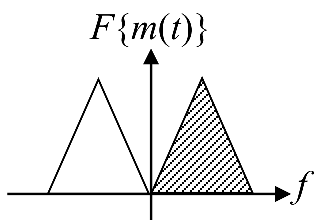
Transmit power $= A_c^2 \overline{m^2} \Rightarrow \dfrac{A_c^2 \overline{m^2}}{N_0 W} = (\mathrm{SNR})_D$ So $(\mathrm{SNR})_D$ equals SNR of a baseband system.

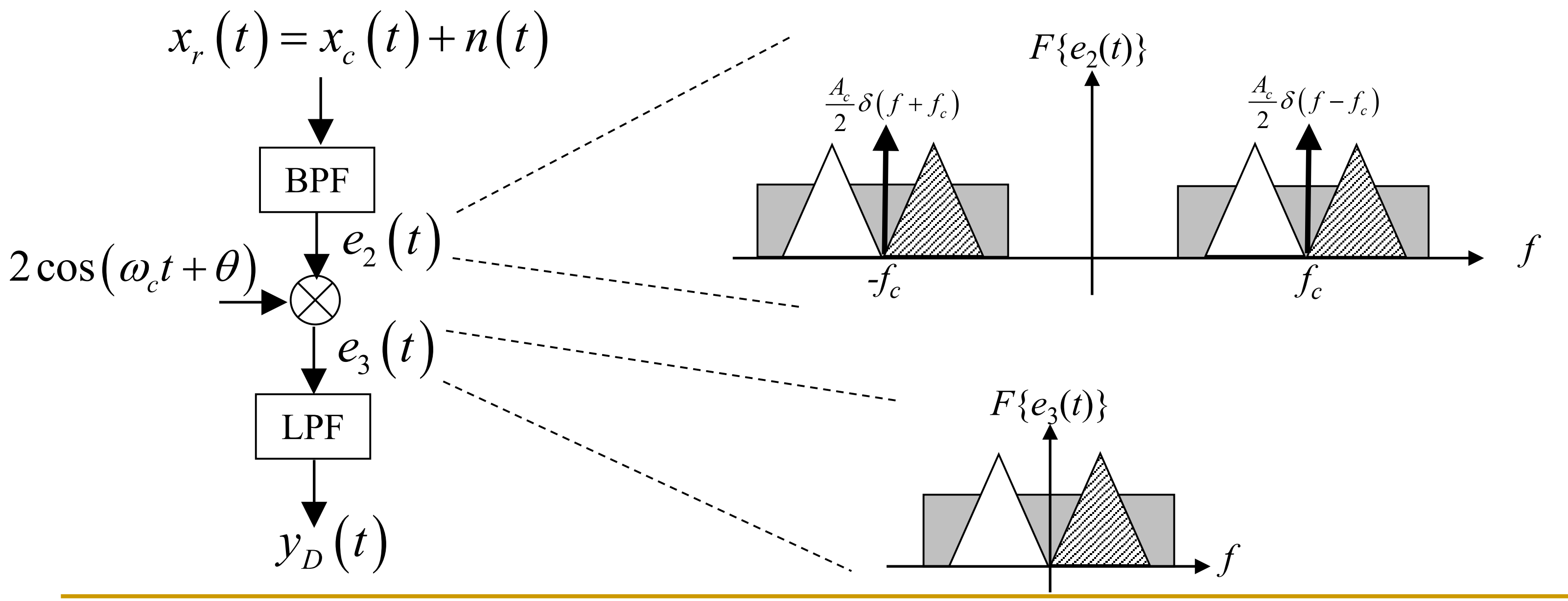
AM System – Coherent Detection

Even though envelope detector (non-coherent detection) is used more often with AM (low implementation complexity), this analysis gives insight into the performance of AM

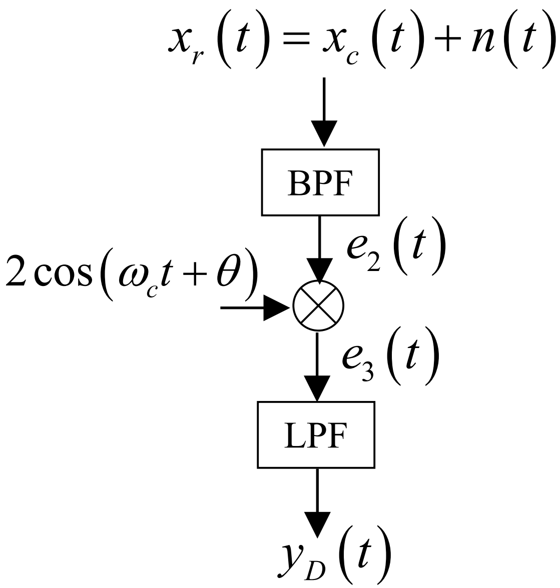
Recall received signal: $x_c(t) = A_c\left[1 + a m_n(t)\right]\cos(\omega_c t + \theta)$,

a : modulation index, m_n : normalized message





AM System – Coherent Detection



(Recall coh. demod. removes quad comp. of $n(t)$ because $\sin(\omega_c t+\theta)\cos(\omega_c t+\theta) = \sin(2\omega_c t + 2\theta)$ so it is removed after LPF)

$$\begin{aligned} y_D(t) &= LP\left\{2A_c\left[1+am_n(t)\right]\cos^2(\omega_c t+\theta)\right\} + n_c(t) \\ &= LP\left\{A_c\left[1+am_n(t)\right] + A_c\left[1+am_n(t)\right]\cos(2\omega_c t + 2\theta)\right\} + n_c(t) \\ &= A_c + A_c a m_n(t) + n_c(t) \end{aligned}$$

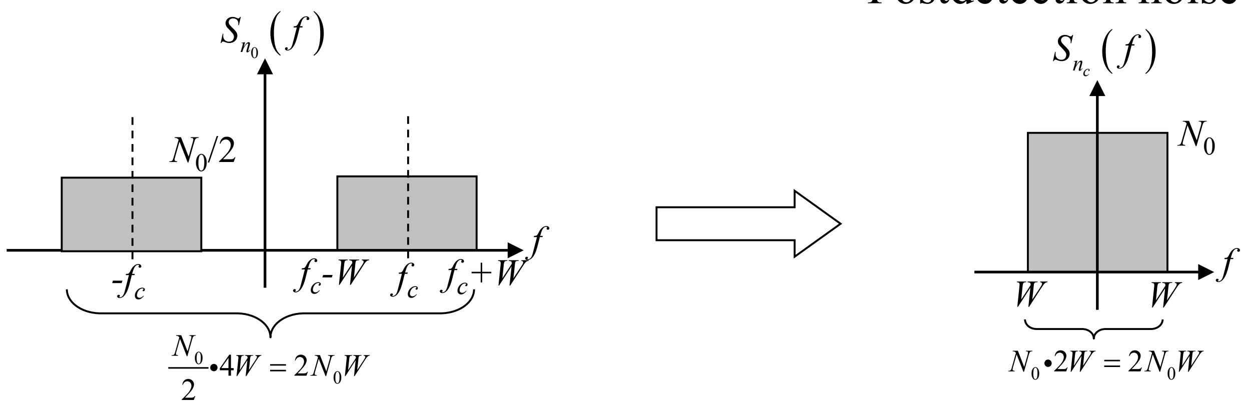
Ignoring DC term $\begin{cases} \text{1) assume } \overline{m(t)} = 0, \text{ we can thus remove DC term} \\ \text{2) In reality, we cannot recover DC term of } m(t), \\ \quad \text{so we simply remove it} \end{cases}$

$$\Rightarrow y_D(t) = A_c a m_n(t) + n_c(t)$$

AM System – Coherent Detection

Predetection noise PSD

Postdetection noise PSD

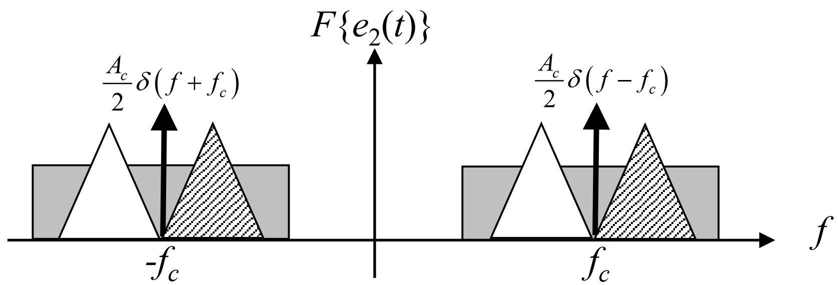


Postdetection signal power: $S_D = \overline{\left(A_c a m_n(t)\right)^2} = A_c^2 a^2 \overline{m_n^2}$

Postdetection noise power: $N_D = \overline{n_c^2} = 2N_0W$

$$\Rightarrow \ (\mathrm{SNR})_D = \frac{A_c^2 a^2 \overline{m_n^2}}{2N_0W}$$

AM System – Coherent Detection



Predetection:

Signal power:

$$S_T = \overline{\left\{A_c\left[1+am_n(t)\right]\cos(\omega_c t+\theta)\right\}^2}$$

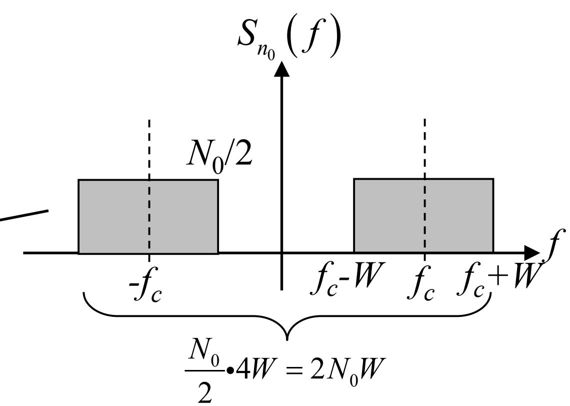
$$= \overline{A_c^2\cos^2(\omega_c t+\theta)} + \overline{A_c^2a^2m_n^2(t)\cos^2(\omega_c t+\theta)}$$

$$= \frac{1}{2}A_c^2 + \frac{1}{2}A_c^2a^2\overline{m_n^2} = P_T$$

BP Noise power: $N_T = 2N_0W$

Predetection SNR: $(\mathrm{SNR})_T = \frac{1}{2}\frac{A_c^2 + A_c^2a^2\overline{m_n^2}}{2N_0W}$

Predetection noise PSD



AM System – Coherent Detection

Detection Gain: $$\frac{(\mathrm{SNR})_D}{(\mathrm{SNR})_T} = \frac{\dfrac{A_c^2 a^2 \overline{m_n^2}}{2N_0W}}{\dfrac{1}{2}\dfrac{A_c^2 + A_c^2 a^2 \overline{m_n^2}}{2N_0W}} = \frac{a^2\overline{m_n^2}}{\frac{1}{2}\left(1+a^2\overline{m_n^2}\right)} = \frac{2a^2\overline{m_n^2}}{1+a^2\overline{m_n^2}} < 1$$

Recall power efficiency: $$E_{ff} = \frac{a^2\overline{m_n^2}}{1+a^2\overline{m_n^2}}$$

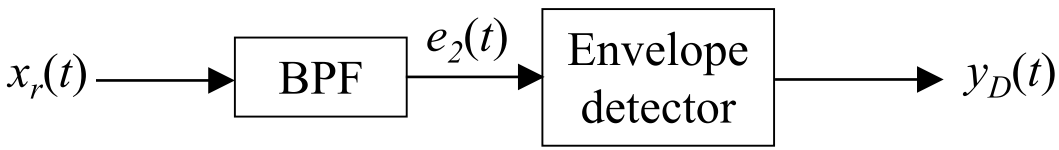
$$\therefore \frac{(\mathrm{SNR})_D}{(\mathrm{SNR})_T} = 2E_{ff}$$

Note: $$(\mathrm{SNR})_D = 2E_{ff} \bullet (\mathrm{SNR})_T = 2E_{ff}\frac{P_T}{2N_0W} = E_{ff}\frac{P_T}{N_0W}$$

Eg. Let $\overline{m_n^2} = 0.1,\quad a = 0.5 \quad \Rightarrow E_{ff} = \dfrac{(0.5)^2\,0.1}{1+(0.5)^2\,0.1} = 0.0244$

$\Rightarrow$ Detection gain $= 2E_{ff} = 0.0488$

AM System – Envelope Detection



$$
\begin{aligned}
e_2(t) &= x_c(t) + n_0(t) \\
&= A_c\left[1 + a m_n(t)\right]\cos(\omega_c t + \theta) + n_c(t)\cos(\omega_c t + \theta) \\
&\quad - n_s(t)\sin(\omega_c t + \theta) \\
&= r(t)\cos\left(\omega_c t + \theta + \phi(t)\right),
\end{aligned}
$$

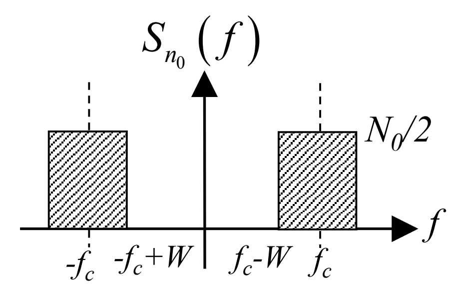
$$
r(t) = \sqrt{\left\{A_c\left[1 + a m_n(t)\right] + n_c(t)\right\}^2 + n_s^2(t)}
$$

$$
\phi(t) = \tan^{-1}\left(\frac{n_s(t)}{A_c\left[1 + a m_n(t)\right] + n_c(t)}\right)
$$

$x_c(t)$: modulated signal

$n_0(t)$: narrowband noise

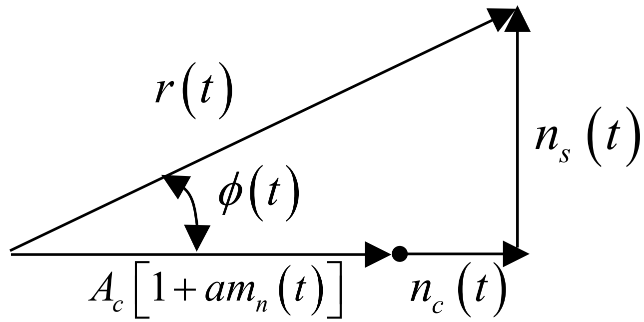
Noise: $\overline{n_c^2} = \overline{n_s^2} = \dfrac{N_0}{2} \cdot 4W = 2N_0 W$



AM System – Envelope Detection

Phasor diagram :

Let $\theta = 0$ (θ does not affect envelope demod)



$$y'_D = r(t) \quad \text{amplitude}$$

Ideally, $\downarrow$ envelope removes DC component (incl. the carrier)

$$y_D(t) = r(t) - \overline{r(t)}, \quad \overline{r(t)}: \text{ average}$$

AM System – Envelope Detection

Case 1: $(\mathrm{SNR})_T$ is large, i.e. small noise

$$\left|A_c\left[1+am_n(t)+n_c(t)\right]\right| \gg n_s(t)$$

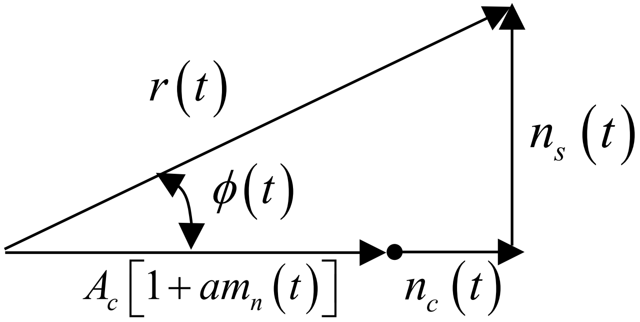
$$\Rightarrow r(t) \approx A_c\left[1+am_n(t)+n_c(t)\right]$$

↑ ↑ zero mean

↑ (after DC is blocked)

$$\Rightarrow y_D(t) \approx A_c a m_n(t)+n_c(t)$$

↑ same as coherent detection - noise is not increased nor decreased



AM System – Envelope Detection

Case 2: $(\mathrm{SNR})_T$ is small, i.e. large noise

Recall $n_c(t)\cos(2\pi f_c t+\theta)-n_s(t)\sin(2\pi f_c t+\theta)$ can be written in

envelope and phase: $n_0(t)=r_n(t)\cos(2\pi f_c t+\phi_n(t))$

So input to envelope detector is

$$e(t)=A_c\left[1+am_n(t)\right]\cos(2\pi f_c t+\theta)+r_n(t)\cos\left[2\pi f_c t+\theta+\phi_n(t)\right]$$

For $(\mathrm{SNR})_T \ll 1$, amplitude of $A_c\left[1+am_n(t)\right] \ll r_n(t)$

From Figure 8.5,

$$\Rightarrow r(t)\approx r_n(t)+A_c\left[1+am_n(t)\right]\cos\left[\phi_n(t)\right]$$

Removing DC term, output of envelope:

$$\Rightarrow y_D(t)\approx r_n(t)+A_c\left[1+am_n(t)\right]\cos\left[\phi_n(t)\right]-\overline{r(t)}$$

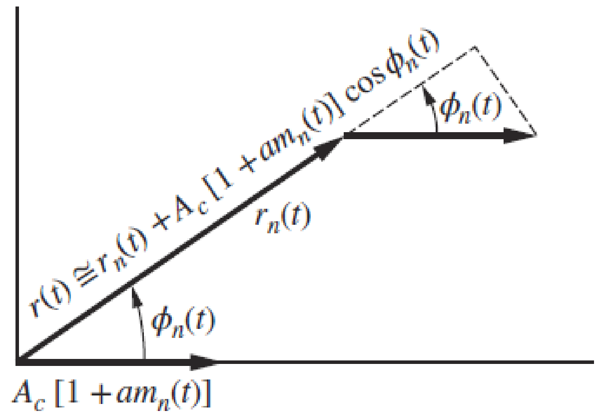


Figure 8.5
Phasor diagram for AM with $(\mathrm{SNR})_T \ll 1$ (drawn for $\theta = 0$).

AM System – Envelope Detection

$$y_D(t) \approx r_n(t) + A_c\left[1 + am_n(t)\right]\cos\left[\phi_n(t)\right] - \overline{r(t)}$$

$\cos\left[\phi_n(t)\right]$: random quantity of noise

$A_c\left[1 + am_n(t)\right]\cos\left[\phi_n(t)\right]$: worsens the signal

This is known as threshold effect because envelope detector is nonlinear, so message is lost when SNR < threshold (don't know what this threshold is)

Note: Difficult to calculate the exact $(\mathrm{SNR})_D$ over wide rangeof SNRs because of square root in the input of the envelope detector:

$$r(t) = \sqrt{\left\{A_c\left[1 + am_n(t)\right] + n_c(t)\right\}^2 + n_s^2(t)} \quad \text{(p. 19)}$$

Definition: The threshold is a value of the carrier-to-noise ratio (or SNR) below which the noise performance of a detector deteriorates much more rapidly than proportionately to the carrier-to-noise ratio (or SNR)

AM System – Envelope Detection

Recall $y_D(t) = A_c a m_n(t) + n_c(t)$ for coherent detection

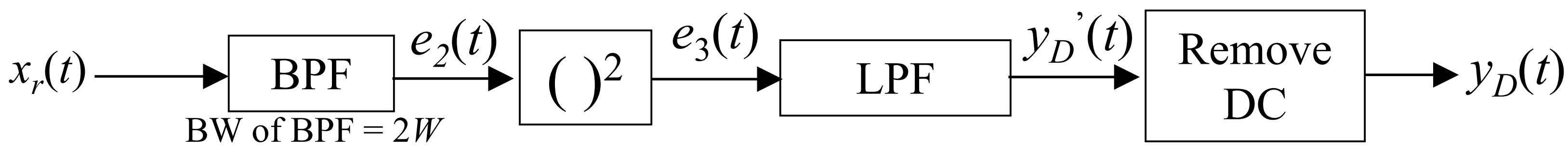
Notice that noise is added to signal at the output of the demodulaor because coherent detectors are linear and the input to it is S+N.

$\Rightarrow$ signal retains its identity even when input SNR is low

$\Rightarrow$ better to use coherent detection when SNR is low

AM System – Square-Law Detection

The square-law detector is an example of simple nonlinear detector that we can calculate SNR and thus threshold region can be more precisely determined



BW of BPF = $2W$

(W = message BW)

DC not a function of signal noise, so block it

$$e_2(t) = x_c(t) + n_0(t),\ \ n_0(t):\ \text{bandpass noise}$$

$$= A_c\left[1 + am_n(t)\right]\cos(2\pi f_c t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

$$e_3(t) = e_2^2(t) = \left\{A_c\left[1 + am_n(t)\right] + n_c(t)\right\}^2 \cos^2(2\pi f_c t)$$

$$-2\left\{A_c\left[1 + am_n(t)\right] + n_c(t)\right\} n_s(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + n_s^2(t)\sin^2(2\pi f_c t)$$

$$= \left\{A_c\left[1 + am_n(t)\right] + n_c(t)\right\}^2 \left[\frac{1}{2} + \frac{1}{2}\cos(4\pi f_c t)\right] - \left\{A_c\left[1 + am_n(t)\right] + n_c(t)\right\} n_s(t)\sin(4\pi f_c t)$$

$$+ n_s^2(t)\left[\frac{1}{2} - \frac{1}{2}\cos(4\pi f_c t)\right]$$

$$\left(\sin(4\pi f_c t), \cos(4\pi f_c t):\ \text{high freq}\right)$$

AM System – Square-Law Detection

$$e_3(t) = \left\{A_c\left[1+am_n(t)\right]+n_c(t)\right\}^2\left[\frac{1}{2}+\frac{1}{2}\cos(4\pi f_c t)\right]-\left\{A_c\left[1+am_n(t)\right]+n_c(t)\right\}n_s(t)\sin(4\pi f_c t)$$
$$+n_s^2(t)\left[\frac{1}{2}-\frac{1}{2}\cos(4\pi f_c t)\right]$$

After LPF:

$$y_D'(t) \propto \left\{A_c\left[1+am_n(t)\right]+n_c(t)\right\}^2+n_s^2(t)$$
$$= r^2(t) \qquad \left(=\text{envelope}^2\right)$$
$$= A_c^2+2A_c^2 am_n(t)+A_c^2a^2m_n^2(t)+2A_cn_c(t)+2A_cam_n(t)n_c(t)+n_c^2(t)+n_s^2(t)$$

After DC removal:

$$y_D(t) = \underbrace{2A_c^2 am_n(t)}_{\text{signal term}}+\underbrace{A_c^2a^2m_n^2(t)}_{\text{signal induced interference}}+\underbrace{2A_cn_c(t)+2A_cam_n(t)n_c(t)+n_c^2(t)+n_s^2(t)}_{\text{Noise terms}}$$

AM System – Square-Law Detection

$$y_D(t) = 2A_c^2 a m_n(t) + A_c^2 a^2 m_n^2(t) + 2A_c n_c(t) + 2A_c a m_n(t) n_c(t) + n_c^2(t) + n_s^2(t)$$

Postdetection signal power: $S_D = 4A_c^4 a^2 \overline{m_n^2(t)}$

Postdetection noise power: $N_D = 4A_c^2 \overline{n_c^2(t)} + 4A_c^2 a^2 \overline{n_c^2(t) m_n^2(t)} + \sigma^2_{n_c^2+n_s^2}$, where

$\sigma^2_{n_c^2+n_s^2} = E\left\{\left[n_c^2(t) + n_s^2(t)\right]^2\right\} - E^2\left[n_c^2(t) + n_s^2(t)\right] = 4\sigma_n^4$ and $\sigma_n^2 = \overline{n^2(t)} = \overline{n_c^2(t)} = \overline{n_s^2(t)}$.

Note also $m_n(t)$ and $\{n_c(t), n_s(t)\}$.

$$\Rightarrow N_D = 4A_c^2\sigma_n^2 + 4A_c^2 a^2 \overline{m_n^2(t)}\sigma_n^2 + 4\sigma_n^4$$

$$\Rightarrow (\text{SNR})_D = \frac{S_D}{N_D} = \frac{4A_c^4 a^2 \overline{m_n^2(t)}}{4A_c^2\sigma_n^2 + 4A_c^2 a^2 \overline{m_n^2(t)}\sigma_n^2 + 4\sigma_n^4} = \frac{A_c^2 A_c^2 a^2 \overline{m_n^2(t)}}{A_c^2\sigma_n^2 + A_c^2 a^2 \overline{m_n^2(t)}\sigma_n^2 + \sigma_n^4}$$

$$= \frac{A_c^2 a^2 \overline{m_n^2(t)}}{\sigma_n^2 + a^2 \overline{m_n^2(t)}\sigma_n^2 + \left(\sigma_n^4 / A_c^2\right)} = \frac{\left(A_c^2/\sigma_n^2\right) a^2 \overline{m_n^2(t)}}{\left[1 + a^2 \overline{m_n^2(t)}\right] + \left(\sigma_n^2 / A_c^2\right)}$$

AM System – Square-Law Detection

$$(\mathrm{SNR})_D = \frac{\left(A_c^2/\sigma_n^2\right)a^2\overline{m_n^2(t)}}{\left[1+a^2\overline{m_n^2(t)}\right]+\left(\sigma_n^2/A_c^2\right)}$$

Recall $P_T = \frac{1}{2}A_c^2\left(1+a^2\overline{m_n^2}\right)$ (p. 17) and (bandpass noise) $\sigma_n^2 = 2N_0W$

$$\Rightarrow \frac{\frac{1}{2}A_c^2\left(1+a^2\overline{m_n^2}\right)}{\sigma_n^2} = \frac{P_T}{2N_0W} \quad \Rightarrow \quad \frac{A_c^2}{\sigma_n^2} = \frac{P_T}{\left(1+a^2\overline{m_n^2}\right)N_0W}$$

$$\Rightarrow (\mathrm{SNR})_D = \frac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)+\left(\sigma_n^2/A_c^2\right)}\frac{P_T}{\left(1+a^2\overline{m_n^2}\right)N_0W} = \frac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2+\left(\sigma_n^2/A_c^2\right)\left(1+a^2\overline{m_n^2}\right)}\frac{P_T/N_0W}{1}$$

$$= \frac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2+\frac{\left(1+a^2\overline{m_n^2}\right)N_0W}{P_T}\left(1+a^2\overline{m_n^2}\right)}\frac{P_T/N_0W}{1} = \frac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2+\frac{\left(1+a^2\overline{m_n^2}\right)^2N_0W}{P_T}}\frac{P_T/N_0W}{1}$$

$$= \frac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2}\frac{P_T/N_0W}{\left(1+N_0W/P_T\right)}$$

AM System – Square-Law Detection

$$(\mathrm{SNR})_D = \frac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2}\frac{P_T/N_0W}{\left(1+N_0W/P_T\right)}$$

$$(\mathrm{SNR})_D = \begin{cases} \dfrac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2}\dfrac{P_T}{N_0W}, & P_T \gg N_0W \quad (N_0W/P_T \approx 0) \qquad \text{(high SNR)} \\[2ex] \dfrac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2}\left(\dfrac{P_T}{N_0W}\right)^2, & P_T \ll N_0W \quad (1+N_0W/P_T \approx N_0W/P_T) \quad \text{(low SNR)} \end{cases}$$

AM System – Square-Law Detection

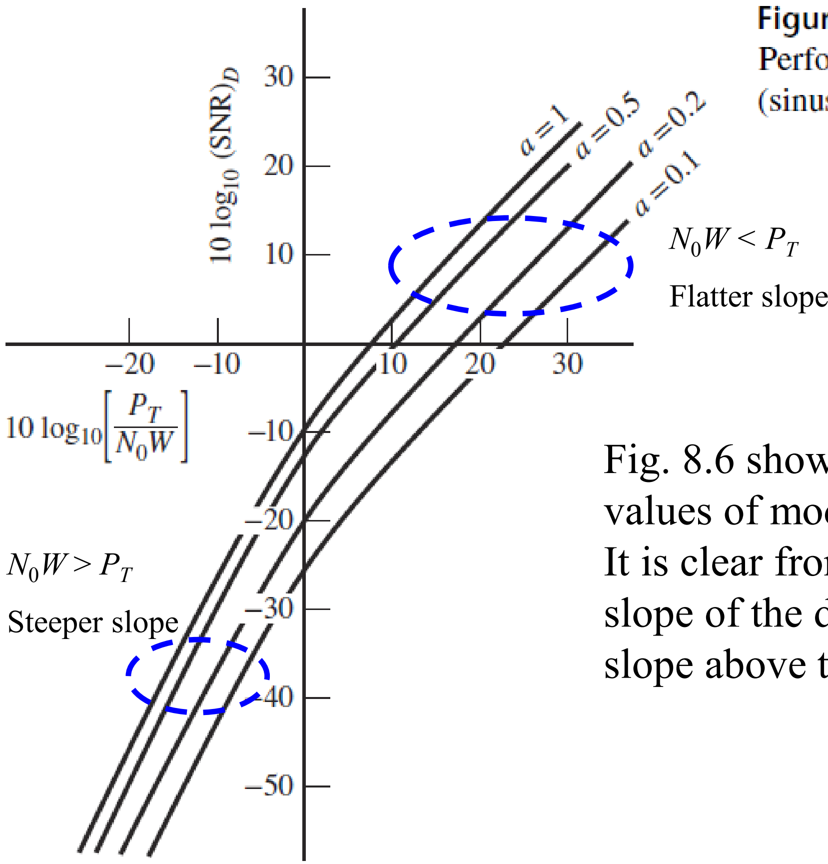


Figure 8.6
Performance of a square-law detector (sinusoidal modulation assumed).

$$(\mathrm{SNR})_D = \frac{a^2\overline{m_n^2}}{\left(1+a^2\overline{m_n^2}\right)^2}\,\frac{P_T/N_0W}{\left(1+N_0W/P_T\right)}$$

Fig. 8.6 shows the $(\mathrm{SNR})_D$ vs. $(\mathrm{SNR})_T$ in dB, for different values of modulation index a.
It is clear from the figure that below some threshold, the slope of the detection gain (i.e. $(\mathrm{SNR})_D$) is double that of the slope above the threshold ➔ threshold effect is obvious

AM System – Square-Law Detection

$$y_D(t) = 2A_c^2 a m_n(t) + A_c^2 a^2 m_n^2(t) + 2A_c n_c(t) + 2A_c a m_n(t) n_c(t) + n_c^2(t) + n_s^2(t)$$

In the derivation of $(\mathrm{SNR})_D$, the signal induced distortion term (2nd term) in $y_D(t)$ was ignored. Define the distortion-to-signal-power ratio as

$$\frac{D_D}{S_D} = \frac{A_c^4 a^4 \overline{m_n^4}}{4A_c^4 a^2 \overline{m_n^2}} = \frac{a^2}{4} \frac{\overline{m_n^4}}{\overline{m_n^2}}.$$

If message is Gaussian with variance σ_m^2, $\overline{m_n^4} = 3\overline{m_n^2}^2 = 3\sigma_m^4$, so

$$\frac{D_D}{S_D} = \frac{3a^2}{4}\sigma_m^2.$$

So the distortion can be reduced by reducing a, but it will also result in reducing $(\mathrm{SNR})_D$

Noise and Phase Errors in Coherent Systems

Random noise and carrier phase error combined effect.

• General signal model: (recall: at the same carrier frequency, we can transmit two "separate" messages - quadrature representation)

$$x_c(t) = m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t$$

(quadrature double-sideband QDSB)

Special cases:

(1) DSB: $\begin{cases} m_1(t) = m(t) \\ m_2(t) = 0 \end{cases}$

(2) SSB: $\begin{cases} m_1(t) = m(t) \\ m_2(t) = \pm\hat{m}(t) \end{cases}$

Assume $\overline{m_1} = \overline{m_2} = 0$

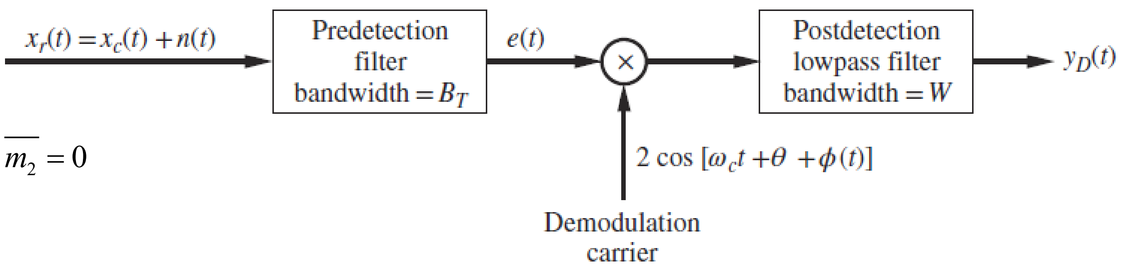


Figure 8.7
Coherent demodulation with phase error.

• Narrowband noise: $\overline{n_c^2} = \overline{n_s^2} = N_0 B_T$

B_T : BW of the predetection filter

$\frac{N_0}{2}$: noise double-sided PSD at predet. filter input

σ_n^2 : noise variance at output of predet. filter

$$n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$

• Receiver local carrier:

$2\cos(\omega_c t + \phi(t))$, $\phi(t)$ - phase error : Gaussian, mean=0, variance=σ_ϕ^2

Note: A constant θ can be inserted at all the above three terms and same results will ensue

Noise and Phase Errors in Coherent Systems

Goal: compute (random) demodulation error statistics $\varepsilon = m_1(t) - y_D(t)$

$$\begin{cases} \overline{\varepsilon} = 0 \\ \text{need to compute } \overline{\varepsilon^2} = E\left\{\left[m_1(t) - y_D(t)\right]^2\right\} \end{cases}$$

(1) $e(t) = m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$

(2) Mult. by $2\cos\left[\omega_c t + \theta + \phi(t)\right]$ and LPF:

$$y_D(t) = \left[m_1(t) + n_c(t)\right]\cos\phi(t) - \left[m_2(t) - n_s(t)\right]\sin\phi(t)$$

(3) Error

$$\varepsilon(t) = m_1(t) - y_D(t) = m_1(t) - \left[m_1(t) + n_c(t)\right]\cos\phi(t) + \left[m_2(t) - n_s(t)\right]\sin\phi(t)$$

$$\overline{\varepsilon^2} = \overline{m_1^2} - 2\overline{m_1(m_1 + n_c)\cos\phi} + 2\overline{m_1(m_2 - n_s)\sin\phi} + \overline{(m_1 + n_c)^2\cos^2\phi}$$
$$-2\overline{(m_1 + n_c)(m_2 - n_s)\sin\phi\cos\phi} + \overline{(m_2 - n_s)^2\sin^2\phi}$$

Assume m_1, m_2, n_c, n_s and ϕ are all uncorrelated

$\therefore\ \overline{m_1(m_1 + n_c)\cos\phi} = \overline{m_1^2\cos\phi} + \overline{m_1 n_c\cos\phi} = \overline{m_1^2}\bullet\overline{\cos\phi}$

(4) $\overline{\varepsilon^2} = \overline{m_1^2} - 2\overline{m_1^2}\bullet\overline{\cos\phi} + \overline{m_1^2}\bullet\overline{\cos^2\phi} + \overline{m_2^2}\bullet\overline{\sin^2\phi} + \overline{n^2}$,

where $\overline{n_c^2} = \overline{n_s^2} = \overline{n^2} = \sigma_n^2 = N_0 B_T$

Noise and Phase Errors in Coherent Systems

Case A: QDSB and $\overline{m_1^2} = \overline{m_2^2} = \sigma_m^2$ (equal power message)

$B_T = 2W$

$$\overline{\varepsilon^2} = \overline{\varepsilon_Q^2} = \overline{m_1^2} - 2\overline{m_1^2} \bullet \overline{\cos\phi} + \overline{m_1^2} \bullet \overline{\cos^2\phi} + \overline{m_2^2} \bullet \overline{\sin^2\phi} + \overline{n^2}$$

$$= \sigma_m^2 - 2\sigma_m^2 \bullet \overline{\cos\phi} + \sigma_m^2 \left(\overline{\cos^2\phi} + \overline{\sin^2\phi} \right) + \sigma_n^2$$

$$= \sigma_m^2 - 2\sigma_m^2 \bullet \overline{\cos\phi} + \sigma_m^2 \left(\frac{1}{2} + \frac{1}{2} \right) + \sigma_n^2$$

$$= 2\sigma_m^2 - 2\sigma_m^2 \bullet \overline{\cos\phi} + \sigma_n^2$$

Assume $|\phi(t)| \ll 1 \Rightarrow \cos\phi(t) \approx 1 - \frac{1}{2}\phi^2(t)$ (1st 2 terms of power series)

$$\Rightarrow \overline{\cos\phi} \approx \overline{1 - \frac{1}{2}\phi^2(t)} = 1 - \frac{1}{2}\sigma_\phi^2$$

$$\Rightarrow \overline{\varepsilon_Q^2} = 2\sigma_m^2 - 2\sigma_m^2 \bullet \overline{\cos\phi} + \sigma_n^2 = 2\sigma_m^2 - 2\sigma_m^2 \bullet \left(1 - \frac{1}{2}\sigma_\phi^2 \right) + \sigma_n^2$$

$$= \underbrace{\sigma_m^2 \sigma_\phi^2}_{\text{multiplicative (phase error)}} + \underbrace{\sigma_n^2}_{\text{additive (white channel noise)}}$$

After normalization: $\overline{\varepsilon_{NQ}^2} = \sigma_\phi^2 + \dfrac{\sigma_n^2}{\sigma_m^2}$

- 2^{nd} term is reciprocal of SNR. When SNR is high, MSE dominated by phase error
- As σ_ϕ^2 decreases, so does the MSE
- As input SNR increases, so does the MSE

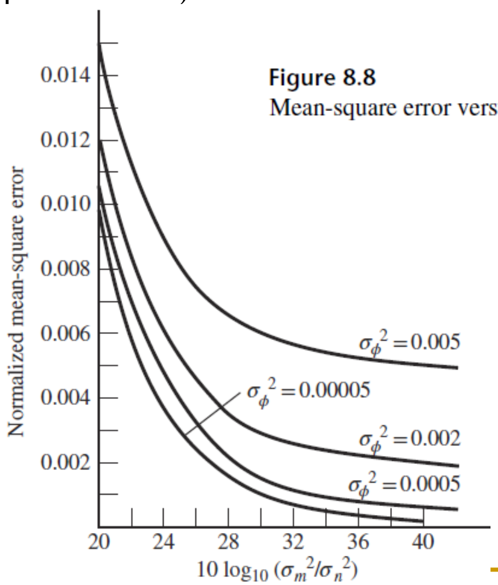


Figure 8.8
Mean-square error versus SNR for QDSB system.

Noise and Phase Errors in Coherent Systems

Case B: DSB and $\begin{cases} m_1(t) = m(t) \\ m_2(t) = 0 \end{cases}$, $B_T = 2W$

$$\overline{m_1^2} = \sigma_m^2$$

$$\overline{\varepsilon_D^2} = \overline{m_1^2} - 2\overline{m_1^2}\bullet\overline{\cos\phi} + \overline{m_1^2}\bullet\overline{\cos^2\phi} + \overline{n^2}$$

$$= \overline{m_1^2}\left(1 - \overline{\cos\phi}\right)^2 + \sigma_n^2$$

Again assume $\left|\phi(t)\right| \ll 1 \;\Rightarrow\; 1 - \cos\phi(t) \approx \frac{1}{2}\phi^2(t)$

$\Rightarrow \overline{\varepsilon_D^2} = \sigma_m^2 \frac{1}{4}\overline{\phi^4} + \sigma_n^2 = \sigma_m^2 \frac{3}{4}\sigma_\phi^4 + \sigma_n^2$ (If ϕ Gaussian and zero mean, $\overline{\phi^4} = \overline{\left(\phi^2\right)^2} = 3\sigma_\phi^4$)

After normalization: $\overline{\varepsilon_{ND}^2} = \frac{3}{4}\sigma_\phi^4 + \frac{\sigma_n^2}{\sigma_m^2}$

Noise and Phase Errors in Coherent Systems

Case C: SSB and $\begin{cases} m_1(t) = m(t) \\ m_2(t) = \pm\hat{m}(t) \end{cases}, \quad B_T = W$

$\overline{m_1^2} = \overline{m_2^2} = \sigma_m^2, \quad m_1(t)$ and $m_2(t)$ are orthogonal

$$\begin{aligned}
\overline{\varepsilon_s^2} &= \overline{m_1^2} - 2\overline{m_1^2}\bullet\overline{\cos\phi} + \overline{m_1^2}\bullet\overline{\cos^2\phi} + \overline{m_2^2}\bullet\overline{\sin^2\phi} + \overline{n^2} \\
&= \sigma_m^2 - 2\sigma_m^2\bullet\overline{\cos\phi} + \sigma_m^2\left(\overline{\cos^2\phi} + \overline{\sin^2\phi}\right) + \sigma_n^2 \\
&= \sigma_m^2 - 2\sigma_m^2\bullet\overline{\cos\phi} + \sigma_m^2\left(\frac{1}{2} + \frac{1}{2}\right) + \sigma_n^2 \\
&= 2\sigma_m^2 - 2\sigma_m^2\bullet\overline{\cos\phi} + \sigma_n^2,
\end{aligned}$$

But $\sigma_n^2 = N_0W \quad (\text{not } 2N_0W)$

Again assume $|\phi(t)| \ll 1 \;\Rightarrow\; \overline{\cos\phi(t)} \approx \overline{1 - \frac{1}{2}\phi^2(t)} = 1 - \frac{1}{2}\sigma_\phi^2$

$$\Rightarrow \overline{\varepsilon_s^2} = 2\sigma_m^2 - 2\sigma_m^2\bullet\overline{\cos\phi} + \sigma_n^2 = 2\sigma_m^2 - 2\sigma_m^2\bullet\left(\overline{1 - \frac{1}{2}\phi^2(t)}\right) + \sigma_n^2 = \sigma_m^2\sigma_\phi^2 + \sigma_n^2$$

After normalization: $\overline{\varepsilon_{NS}^2} = \sigma_\phi^2 + \dfrac{\sigma_n^2}{\sigma_m^2} \quad \Rightarrow$ same result as QDSB

Noise and Phase Errors in Coherent Systems

MSE

SSB: $\overline{\varepsilon_{NS}^2} = \sigma_\phi^2 + \dfrac{\sigma_n^2}{\sigma_m^2}$

DSB: $\overline{\varepsilon_{ND}^2} = \dfrac{3}{4}\sigma_\phi^4 + \dfrac{\sigma_n^2}{\sigma_m^2}$

Remarks :

(1) $\sigma_\phi^2 = 0$ (no carrier phase error)

$$\Rightarrow \overline{\varepsilon_{N*}^2} = \frac{\sigma_n^2}{\sigma_m^2} = \text{coherent detection SNR}$$

(2) DSB is less sensitive to phase error than SSB or QDSB

because if $|\phi(t)| \ll 1$ (basic assumption made in above calculation),

$\Rightarrow \sigma_\phi^4 \ll \sigma_\phi^2$

Noise in Angle Modulation

Recall the BW for angle modulation can be approximated using Carson's rule

$BW \quad B_T = 2(D+1)W$ Hz

$$D = \begin{cases} \dfrac{\beta f_m}{W} & \text{(PM)} \\ \dfrac{Af_d}{W} = \beta & \text{(FM)} \end{cases} : \text{peak deviation}, \quad W = \text{message BW}$$

(for PM, $\beta = Ak_p$)

Received signal

$$x_r(t) = x_c(t) + n(t) = A_c \cos\left[\omega_c t + \phi(t)\right] + n(t),$$

$$\phi(t) = \begin{cases} k_p m_n(t), & \text{for PM} \\ k_f \int^t m_n(\alpha)\, d\alpha = 2\pi f_d \int^t m_n(\alpha)\, d\alpha, & \text{for FM} \end{cases} \text{ is the phase deviation}$$

$n(t)$: AWGN with PSD amplitude $\dfrac{N_0}{2}$ (W/Hz)

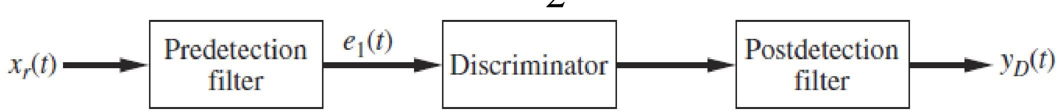


Figure 8.9
Angle demodulation system.

Noise in Angle Modulation

Output of the predetection filter:

$$e_1(t) = x_c(t) + n(t) = A_c \cos\left[\omega_c t + \phi(t)\right] + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$

$$= A_c \cos\left[\omega_c t + \phi(t)\right] + r_n(t)\cos\left[\omega_c t + \phi_n(t)\right]$$

$r_n(t)$: Rayleigh distribution

$\phi_n(t)$: uniform distribution

$$\overline{n_c^2} = \overline{n_c^2} = N_0 B_T$$

Let $\phi_n(t) + \omega_c t = \phi_n(t) - \phi(t) + \omega_c t + \phi(t)$

$$e_1(t) = A_c \cos\left[\omega_c t + \phi(t)\right] + r_n(t)\cos\left[\phi_n(t) - \phi(t) + \omega_c t + \phi(t)\right]$$

$$= A_c \cos\left[\omega_c t + \phi(t)\right] + r_n(t)\cos\left[\phi_n(t) - \phi(t)\right]\cos\left[\omega_c t + \phi(t)\right]$$

$$- r_n(t)\sin\left[\phi_n(t) - \phi(t)\right]\sin\left[\omega_c t + \phi(t)\right]$$

$$= \left\{A_c + r_n(t)\cos\left[\phi_n(t) - \phi(t)\right]\right\}\cos\left[\omega_c t + \phi(t)\right] - r_n(t)\sin\left[\phi_n(t) - \phi(t)\right]\sin\left[\omega_c t + \phi(t)\right]$$

$$= R(t)\cos\left[\omega_c t + \phi(t) + \phi_e(t)\right],$$

$\phi_e(t) = \tan^{-1}\left(\dfrac{r_n(t)\sin\left[\phi_n(t) - \phi(t)\right]}{A_c + r_n(t)\cos\left[\phi_n(t) - \phi(t)\right]}\right)$ is phase deviation error due to noise

Noise in Angle Modulation

$$e_1(t) = R(t)\cos\left[\omega_c t + \phi(t) + \phi_e(t)\right],$$

$$R(t) = \sqrt{\left\{A_c + r_n(t)\cos\left[\phi_n(t) - \phi(t)\right]\right\}^2 + \left\{r_n(t)\sin\left[\phi_n(t) - \phi(t)\right]\right\}^2}$$

$$\psi(t) = \phi(t) + \phi_e(t)\begin{cases}\text{demod output is proportional to } \psi(t) \text{ in PM} \\ \text{demod output proportional to } \dfrac{d\psi(t)}{dt} \text{ in FM}\end{cases}$$

is the phase deviation of the discriminator input due to the combination of signal and noise.

Since demodulated output is different for PM and FM, must consider them separately.

$$\Rightarrow e_1(t) = R(t)\cos\left[\omega_c t + \psi(t)\right]$$

(If $\phi_e(t) = 0$, perfect recovery of message)

Noise in Angle Modulation

Case A: High predetection $(\mathrm{SNR})_T$ or $A_c \gg r_n(t)$

$$\phi_e(t) = \tan^{-1}\left(\frac{r_n(t)\sin[\phi_n(t)-\phi(t)]}{A_c + r_n(t)\cos[\phi_n(t)-\phi(t)]}\right) \approx \frac{r_n(t)}{A_c}\sin[\phi_n(t)-\phi(t)]$$

$$\Rightarrow \psi(t) = \phi(t) + \frac{r_n(t)}{A_c}\sin[\phi_n(t)-\phi(t)]$$

$$\because \phi_n(t) \sim \mathrm{Unif}[0,2\pi),\ \text{and } \phi(t) = \text{constant at given } t$$

$$\Rightarrow \phi_n(t) - \phi(t) = \mathrm{mod}(2\pi) \Rightarrow \text{neglect } \phi(t)$$

$$\Rightarrow \psi(t) = \phi(t) + \frac{r_n(t)}{A_c}\sin[\phi_n(t)]$$

$$\because \ \sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{r_n(t)}{A_c}\sin[\phi_n(t)] = \frac{r_n(t)}{A_c}\frac{n_s(t)/n_c(t)}{\sqrt{1+[n_s(t)/n_c(t)]^2}}$$

$$= \frac{r_n(t)}{A_c}\frac{n_s(t)}{\sqrt{n_c^2(t)+n_s^2(t)}} = \frac{n_s(t)}{A_c}$$

$$\Rightarrow \psi(t) = \phi(t) + \frac{n_s(t)}{A_c} \quad \text{(S+N - additive noise model)}$$

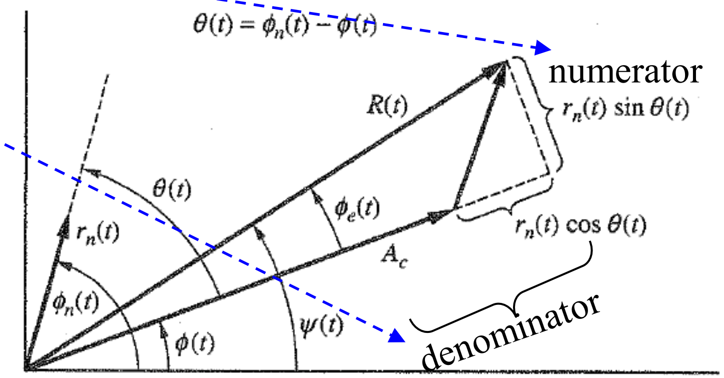


FIGURE 6.12 Phasor diagram for angle demodulation, assuming $(\mathrm{SNR})_T > 1$ (drawn for $\theta = 0$).

Noise in Angle Modulation

Recall output of the discriminator (p. 82 of lec03A)

$$y_D(t) = \begin{cases} K_D\phi(t) = K_D k_p m_n(t), & \text{PM} \\ \dfrac{1}{2\pi}K_D \dfrac{d\phi(t)}{dt} = \dfrac{1}{2\pi}K_D \dfrac{d}{dt}2\pi f_d \int^t m_n(\alpha)\,d\alpha & \\ \qquad = K_D f_d m_n(t), & \text{FM} \end{cases},$$

K_D : discriminator constant (volts/Hz)

With noise, the output becomes

$$y_D(t) = \begin{cases} K_D\psi(t), & \text{PM} \\ \dfrac{1}{2\pi}K_D \dfrac{d\psi(t)}{dt}, & \text{FM} \end{cases}$$

$$\psi(t) = \phi(t) + \frac{n_s(t)}{A_c}$$

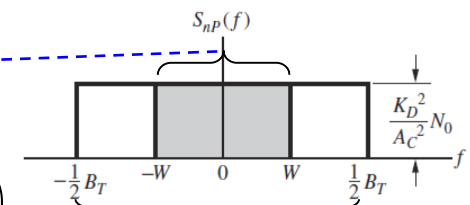
Noise in Phase Modulation

(A.1) For PM: $y_{DP}(t) = K_D\phi(t) + K_D\dfrac{n_s(t)}{A_c} = K_D k_p m_n(t) + K_D\dfrac{n_s(t)}{A_c}$

Postdet filter output signal power: $S_{DP} = K_D^2 k_p^2 \overline{m_n^2}$

Postdet filter output Noise power: $\dfrac{K_D^2}{A_c^2}\overline{n_s^2} = \dfrac{K_D^2}{A_c^2}\int_{-W}^{W} N_0 df$ (using a LPF to reduce $B_T > 2W \to 2W$)

$$= 2\frac{K_D^2}{A_c^2} N_0 W$$



$\left(\text{Note that input noise power to discriminator: } \dfrac{K_D^2}{A_c^2}\int_{-B_T/2}^{B_T/2} N_0 df = \dfrac{K_D^2}{A_c^2} N_0 B_T\right)$

PSD of PM discriminator output

$$\Rightarrow (\text{SNR})_D = \frac{A_c^2 k_p^2 \overline{m_n^2}}{2N_0 W}$$

Recall the info bearing signal of $e_1(t)$ is $A_c\cos\left[\omega_c t + \phi(t)\right] \Rightarrow P_T = \overline{\left\{A_c\cos\left[\omega_c t + \phi(t)\right]\right\}^2} = \dfrac{A_c^2}{2}$

$$\Rightarrow (\text{SNR})_D = k_p^2 \overline{m_n^2}\frac{P_T}{N_0 W}$$

Noise in Phase Modulation

$$(\mathrm{SNR})_D = k_p^2 \overline{m_n^2} \frac{P_T}{N_0 W}$$

In order to assure unique demodulation

$$\left|k_p m_n(t)\right| = \text{phase deviation } \le \pi$$

Unique demodulation of PM signal cannot be accomplished when phase deviation is greater than π

$$\Rightarrow k_p^2 \overline{m_n^2} \le \pi^2 \Rightarrow 10\log_{10}\pi^2 \approx 10 \text{ dB!}$$

Since baseband SNR $= \dfrac{P_T}{N_0 W}$

The maximum improvement of the SNR = 10.

Also compared this to (high input SNR) AM (square law)

$$(\mathrm{SNR})_D \approx \frac{a^2 \overline{m_n^2(t)}}{\left(1 + a^2 \overline{m_n^2(t)}\right)^2} \frac{P_T}{N_0 W} = (<1) \frac{P_T}{N_0 W}$$

$\therefore$ PM has higher $(\mathrm{SNR})_D$

Noise in Frequency Modulation

$$\psi(t) = \phi(t) + \frac{n_s(t)}{A_c}$$

(A.2) FM $y_{DF}(t) = \frac{1}{2\pi} K_D \frac{d\psi(t)}{dt} = \frac{1}{2\pi} K_D \frac{d\phi(t)}{dt} + \frac{K_D}{2\pi A_c} \frac{dn_s(t)}{dt}$

$$= K_D f_d m_n(t) + \frac{K_D}{2\pi A_c} \frac{dn_s(t)}{dt}$$

Postdet. filter output signal power: $S_{DF} = K_D^2 f_d^2 \overline{m_n^2}$

Postdet. filter output noise power:

First, define $n_F(t) \triangleq \frac{K_D}{2\pi A_c} \frac{dn_s(t)}{dt}$ as noise signal at demodulator output

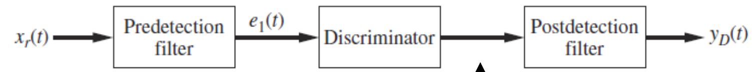


Figure 8.9
Angle demodulation system.

$$\Rightarrow S_{nF}(f) = \frac{K_D^2}{(2\pi)^2 A_c^2} (2\pi f)^2 N_0 = \frac{K_D^2}{A_c^2} N_0 f^2$$

$$\left(\because\ S_y(f) = |H(f)|^2 S_x(f), \text{ with } S_x(f) = N_0,\ H(f) = j2\pi f \Leftrightarrow \frac{d}{dt} \right)$$

Noise spectrum results from use of differentiator

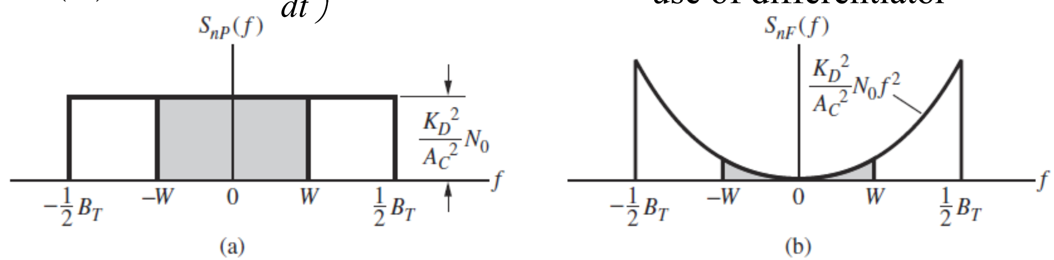


Figure 8.10
(a) Power spectral density for PM discriminator output with portion for $|f| < W$ shaded. (b) Power spectral density for FM discriminator output with portion for $|f| < W$ shaded.

Noise in Frequency Modulation

At the postdetection filter output (LPF with $2W$ BW)

$$N_{DF} = \frac{K_D^2}{A_c^2} N_0 \int_{-W}^{W} f^2 df = \frac{2}{3} \frac{K_D^2}{A_c^2} N_0 W^3$$

Recall signal power of $e_1(t) \triangleq P_T = \frac{A_c^2}{2} \Rightarrow \frac{P_T}{N_0 W} = \frac{A_c^2}{2 N_0 W}$ (just like PM, write $(\mathrm{SNR})_D$ in terms of this ratio)

$$\Rightarrow N_{DF} = \frac{1}{3} K_D^2 W^2 \left(\frac{P_T}{N_0 W} \right)^{-1}$$

So, postdetection SNR

$$\Rightarrow (\mathrm{SNR})_{DF} = \frac{S_{DF}}{N_{DF}} = \frac{K_D^2 f_d^2 \overline{m_n^2}}{\frac{1}{3} K_D^2 W^2 \left(\frac{P_T}{N_0 W} \right)^{-1}} = 3 \left(\frac{f_d}{W} \right)^2 \overline{m_n^2} \frac{P_T}{N_0 W}$$

This analysis is based on the assumption that $A_c >> r_n(t)$, or high $(\mathrm{SNR})_T$ or "above threshold"

Recall for FM: $D = \frac{f_d A}{W}$, assuming $A = 1$,

$$\Rightarrow (\mathrm{SNR})_{DF} = 3 D^2 \overline{m_n^2} \frac{P_T}{N_0 W}$$

Since $\frac{d\phi(t)}{dt} \approx 2\pi f_d m_n(t)$ and $|m_n(t)| \le 1$

$\therefore$ $(\mathrm{SNR})_D$ improvement $\approx 3D^2 \overline{m_n^2}$ over baseband system

$\Rightarrow$ if $D \uparrow$ $\Rightarrow$ $(\mathrm{SNR})_D \uparrow$ $\Rightarrow$ transmit BW $\uparrow$ (from cf. Carson's rule) $\leftarrow$ So, cannot increase D indefinitely

Noise in Angle Modulation (Low $(\text{SNR})_T$)

Case B: Low predetection $(\text{SNR})_T$ or $A_c \ll r_n(t)$

Recall output of predet. filter

$$e_1(t) = R(t)\cos\big[\omega_c t + \overbrace{\phi(t)+\phi_e(t)}^{\psi(t)}\big],$$

$$\psi(t) = \phi(t)+\phi_e(t) = \begin{cases} \text{proportional to PM} \\ \dfrac{d\psi(t)}{dt} \ \text{prop. to FM} \end{cases},$$

$$\Rightarrow e_1(t) = R(t)\cos\big[\omega_c t + \psi(t)\big],$$

$$\text{with } R(t) = \sqrt{\big(A_c + r_n(t)\cos[\phi_n(t)-\phi(t)]\big)^2 + \big(r_n(t)\sin[\phi_n(t)-\phi(t)]\big)^2}$$

$$= \sqrt{A_c^2 + 2A_c r_n(t)\cos[\phi_n(t)-\phi(t)] + r_n^2(t)}$$

$$\approx r_n(t)$$

$$\Rightarrow A_c \sin[\phi_n(t)-\phi(t)] = R(t)\sin[\phi_n(t)-\psi(t)] \approx r_n(t)[\phi_n(t)-\psi(t)],$$

for $[\phi_n(t)-\psi(t)]$ small. Solving for $\psi(t)$,

$$\Rightarrow \psi(t) \triangleq \phi_n(t) - \frac{A_c}{r_n(t)}\sin[\phi_n(t)-\phi(t)],$$

$\psi(t)$ is dominated by noise; message is lost. This is the threshold effect.

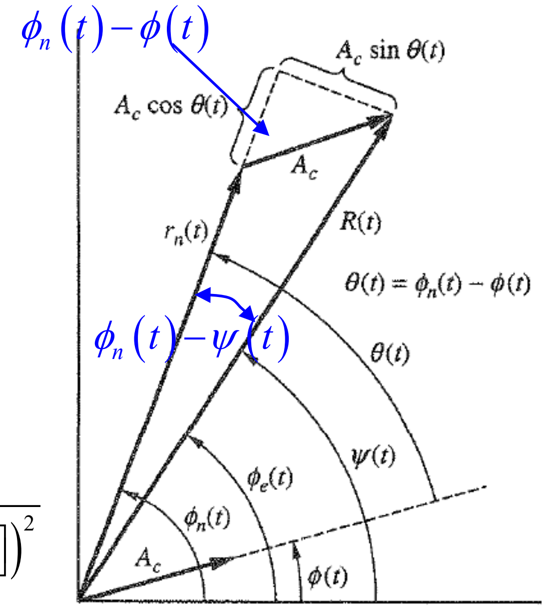


FIGURE 6.13 Phasor diagram for angle demodulation, assuming $(\text{SNR})_T < 1$ (drawn for $\theta = 0$).

p. 39

$$e_1(t) = \{A_c + r_n(t)\cos[\phi_n(t)-\phi(t)]\}\cos[\omega_c t + \phi(t)]$$
$$- r_n(t)\sin[\phi_n(t)-\phi(t)]\sin[\omega_c t + \phi(t)]$$

Pre-Emphasis and De-emphasis Filter

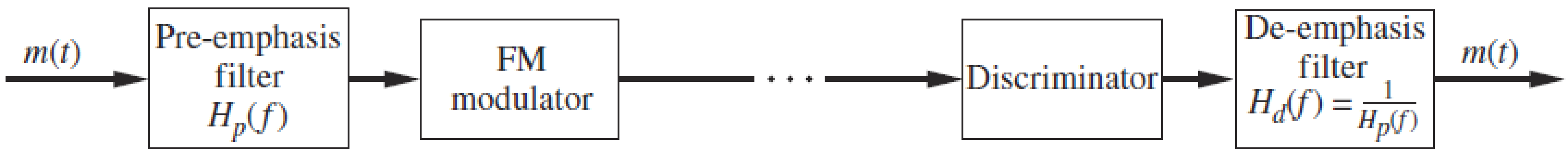


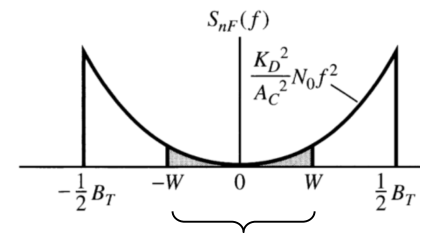
Figure 4.32
Frequency modulation system with pre-emphasis and de-emphasis.

Let Deemphasis filter $= H_{DE}(f)$ (usu. a LPF)

Preemphasis filter $= \dfrac{1}{H_{DE}(f)}$

Why pre- and deemphasis?

Change the shape of message spectrum before modulation to take advantage of the nonlinear characteristics of FM noise spectrum



Low frequency of demod output has less noise in FM because of the use of the differentiator as demodulator

Pre-Emphasis and De-emphasis Filter

Assume: $H_{DE}(f)=\dfrac{1}{1+j\dfrac{f}{f_{3\text{dB}}}}$, $\quad f_{3\text{dB}}=\dfrac{1}{2\pi RC} \quad\Rightarrow\quad \left|H_{DE}(f)\right|=\dfrac{1}{\sqrt{1+\left(\dfrac{f}{f_{3\text{dB}}}\right)^2}}$

De-emphasis filter output noise power:

$$N_{DF}=\int_{-W}^{W}\left|H_{DE}(f)\right|^2 S_{nF}(f)\,df=\int_{-W}^{W}\frac{1}{\sqrt{1+\left(\dfrac{f}{f_{3\text{dB}}}\right)^2}}\frac{K_D^2}{A_c^2}N_0 f^2 df$$

$$=2\frac{K_D^2}{A_c^2}N_0 f_{3\text{dB}}^3\left(\frac{W}{f_{3\text{dB}}}-\tan^{-1}\frac{W}{f_{3\text{dB}}}\right)$$

Figure 4.32
Frequency modulation system with pre-emphasis and de-emphasis.

Typically, $f_{3\text{dB}} \ll W$, so $\tan^{-1}\dfrac{W}{f_{3\text{dB}}}\approx\dfrac{\pi}{2}<\dfrac{W}{f_{3\text{dB}}}$

$$\Rightarrow N_{DF}=2\frac{K_D^2}{A_c^2}N_0 f_{3\text{dB}}^2 W=K_D^2 f_{3\text{dB}}^2\left(\frac{P_T}{N_0W}\right)^{-1} \qquad \left(\text{signal power of } e_1(t)\triangleq P_T=\frac{A_c^2}{2}\right)$$

$\Rightarrow$ De-emphassis filter output SNR:

$$(\text{SNR})_{DF}=\frac{S_{DF}}{N_{DF}}=\frac{K_D^2 f_d^2\overline{m_n^2}}{K_D^2 f_{3\text{dB}}^2\left(\dfrac{P_T}{N_0W}\right)^{-1}}=\left(\frac{f_d}{f_{3\text{dB}}}\right)^2\overline{m_n^2}\frac{P_T}{N_0W}$$

Without emphasis, postdet filt. out SNR:

$$(\text{SNR})_{DF}=3\left(\frac{f_d}{W}\right)^2\overline{m_n^2}\frac{P_T}{N_0W}$$

So, if $f_{3\text{dB}} \ll W$, improvement using pre- and de-emphasis filters is significant.

Example 8.3

Commerical FM operates with $f_d = 75$ kHz, $W = 15$ kHz, $D = 5$. Assume $\overline{m_n^2} = 0.1$

(1) w/o pre- and de-emphasis:

$$(\text{SNR})_{DF} = 3\left(\frac{f_d}{W}\right)^2 \overline{m_n^2} \frac{P_T}{N_0 W} = 3\left(\frac{75}{15}\right)^2 0.1 \frac{P_T}{N_0 W} = 7.5 \frac{P_T}{N_0 W}$$

(1) w/ pre- and de-emphasis, and $f_{3\text{dB}} = 2.1$ kHz:

$$(\text{SNR})_{DF} = \left(\frac{f_d}{f_{3\text{dB}}}\right)^2 \overline{m_n^2} \frac{P_T}{N_0 W} = \left(\frac{75}{2.1}\right)^2 0.1 \frac{P_T}{N_0 W} = 128 \frac{P_T}{N_0 W}$$

Remark: Disadvantage of de-emphasis: may increase transmission BW because pre-emphasis filter is a HP, so it amplifies high freq portion of message signal, thus increasing frequency deviation $D \Rightarrow$ transmission $BW \uparrow$.

However, typlical message sources have "little" high frequency components so effect is negligible.

Threshold Effect in FM Demodulation

- For FM modulation, signal is embedded in $\theta(t)$
- Threshold effect can be studied by looking at the rate of change of the phase
- This can be obtained by observing the output of the discriminator (demodulator) with input being an unmodulated sinusoid + additive bandlimited white noise
  - As $(\text{SNR})_T$ is gradually decreased, more spikes will occur, thus giving us a way to compute the PSD
  - Will see that modulation will increase the rate of spikes
    - Additive can be modeled as:

      Gaussian noise (not a function of modulation)

      + Spike noise (function of modulation)
- Also will like to know the statistics of the number of spikes in a given time interval
  - Allows for computation of its PSD
    - Allows for computation of alternate expression for $(\text{SNR})_D$ where the threshold effect is more transparent

First sign of spikes denotes discriminator is operating in threshold region

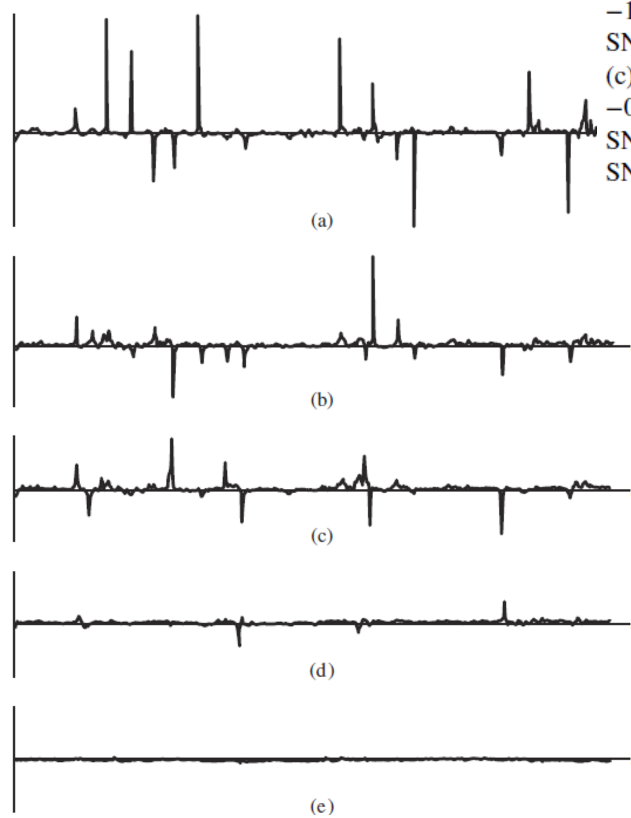


Figure 8.13
Output of FM discriminator due to input noise for various predetection SNRs.
(a) Predetection SNR = −10 dB. (b) Predetection SNR = −4 dB.
(c) Predetection SNR = −0 dB. (d) Predetection SNR = 6 dB. (e) Predetection SNR = 10 dB.

Threshold Effect in FM Demodulation

Probability of an PSD spike noise

- Zero-crossing probability

Assume a lowpass, zero-mean white Gaussian noise $n(t)$

with BW $= W$, PSD $S_n(f) \leftrightarrow$ autocorrelation $R_n(\tau)$

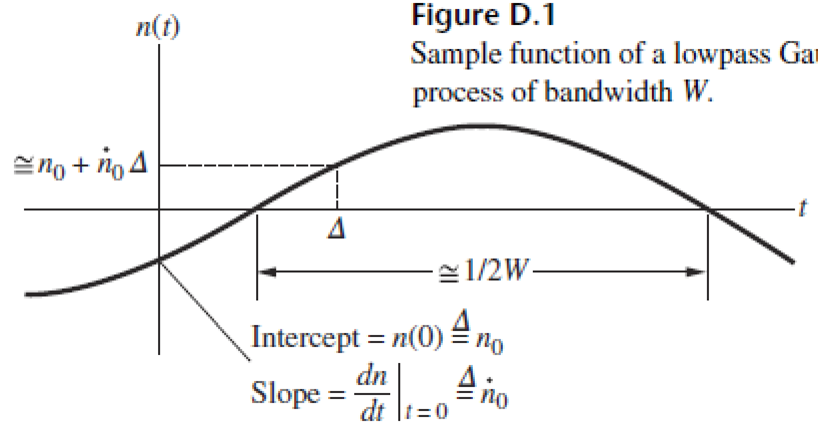
Consider the probability of a zero crossing in a small time interval Δ seconds. When Δ is small, the probability of having more than one zero crossing is unlikely.

$P_{\Delta-}$: Prob. of minus-to-plus zero crossing in $\Delta \ll \dfrac{1}{2W}$ seconds equals the probability

that $n_0 < 0$ and $n_0 + n_0'\Delta > 0$

$$
\begin{aligned}
P_{\Delta-} &= \Pr\left(n_0 < 0 \text{ and } \left(n_0 + n_0'\Delta\right) > 0\right) \\
&= \Pr\left(n_0 < 0 \text{ and } n_0 > -n_0'\Delta\right) \\
&= \Pr\left(-n_0'\Delta < n_0 < 0\right)
\end{aligned}
$$

Figure D.1
Sample function of a lowpass Gaussian process of bandwidth W.



Threshold Effect in FM Demodulation

joint PDF of (n_0, n_0'): $f(n_0, n_0') = f_{n_0 n_0'}(y,z)$

$P_{\Delta-} = \int_0^{\infty}\left[\int_{-z\Delta}^{0} f_{n_0 n_0'}(y,z)\,dy\right]dz$ (note that n_0' is also Gaussian as $\frac{d}{dt}$ is a linear operation)

Since it can shown that $E[n_0 n_0'] = \left.\frac{dR_n(\tau)}{d\tau}\right|_{\tau=0} = 0 \Rightarrow n_0$ and n_0' uncorrelated.

Since $n(t)$ Gaussian, so is n_0', and so n_0 and n_0' stat. independent.

$$\Rightarrow f_{n_0 n_0'}(y,z) = f_{n_0}(y) f_{n_0'}(z) = \frac{1}{\sqrt{2\pi \overline{\overline{n_0^2}}}}\exp\left(-\frac{y^2}{2\overline{\overline{n_0^2}}}\right)\frac{1}{\sqrt{2\pi \overline{\overline{n_0'^2}}}}\exp\left(-\frac{z^2}{2\overline{\overline{n_0'^2}}}\right)$$

$$\Rightarrow P_{\Delta-} = \int_0^{\infty}\left[\int_{-z\Delta}^{0} f_{n_0 n_0'}(y,z)\,dy\right]dz$$

$$= \int_0^{\infty}\frac{1}{\sqrt{2\pi \overline{\overline{n_0'^2}}}}\exp\left(-\frac{z^2}{2\overline{\overline{n_0'^2}}}\right)\left[\int_{-z\Delta}^{0}\frac{1}{\sqrt{2\pi \overline{\overline{n_0^2}}}}\exp\left(-\frac{y^2}{2\overline{\overline{n_0^2}}}\right)dy\right]dz$$

Threshold Effect in FM Demodulation

$$P_{\Delta-} = \int_0^\infty \left[\int_{-z\Delta}^0 f_{n_0 n_0'}(y,z)\,dy \right] dz$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi \overline{n_0'^2}}} \exp\left(-\frac{z^2}{2\overline{n_0'^2}}\right) \left[\int_{-z\Delta}^0 \frac{1}{\sqrt{2\pi \overline{n_0^2}}} \exp\left(-\frac{y^2}{2\overline{n_0^2}}\right) dy \right] dz$$

(if Δ small, inner integral can be approximated as $\frac{\Delta z}{\sqrt{2\pi \overline{n_0^2}}}$)

$$P_{\Delta-} \approx \frac{\Delta}{\sqrt{2\pi \overline{n_0^2}}} \int_0^\infty \frac{z}{\sqrt{2\pi \overline{n_0'^2}}} \exp\left(-\frac{z^2}{2\overline{n_0'^2}}\right) dz = \frac{\Delta}{2\pi\sqrt{\overline{n_0^2}\,\overline{n_0'^2}}} \int_0^\infty z \exp\left(-\frac{z^2}{2\overline{n_0'^2}}\right) dz$$

Let $\zeta = \frac{z^2}{2\overline{n_0'^2}}, \quad d\zeta = \frac{2z}{2\overline{n_0'^2}} dz = \frac{z}{\overline{n_0'^2}} dz$

$$P_{\Delta-} \approx \frac{\Delta}{\sqrt{2\pi \overline{n_0^2}\,\overline{n_0'^2}}} \int_0^\infty \overline{n_0'^2} e^{-\zeta} d\zeta = \frac{\Delta}{2\pi} \sqrt{\frac{\overline{n_0'^2}}{\overline{n_0^2}}}$$

(which is the minus-to-plus zero crossing in Δ seconds)

Threshold Effect in FM Demodulation

Example: suppose $n(t)$ is an ideal lowpass process with PSD

$$S_n(f)=\begin{cases}\dfrac{N_0}{2}, & |f|\le W\\ 0, & \text{otherwise}\end{cases} \quad\Leftrightarrow\quad R_n(\tau)=N_0W\operatorname{sinc}2W\tau.$$

Note: $\overline{n_0^2}=\operatorname{var}[n_0]=\int_{-\infty}^{\infty}S_n(f)\,df=R_n(0)=N_0W.$

Note $E[n_0n_0']=\left.\dfrac{dR_n(\tau)}{d\tau}\right|_{\tau=0}=0 \;\Rightarrow\; n_0$ and n_0' stat. independent.

$$\overline{n_0^2}=\operatorname{var}(n_0)=\int_f S_n(f)\,df=R_n(0)=N_0W$$

Let $H_d(f)=j2\pi f$: freq response of ideal differentiator

$$\overline{n_0'^2}=\operatorname{var}(n_0')=\int_f |H_d(f)|^2 S_n(f)\,df=\int_{-W}^{W}(2\pi f)^2\frac{N_0}{2}df=\frac{1}{3}(2\pi W)^2 N_0W$$

Since $P_{\Delta-}=P_{\Delta+}\approx\dfrac{\Delta}{2\pi}\sqrt{\dfrac{\overline{n_0'^2}}{\overline{n_0^2}}} \;\Rightarrow\; P_\Delta=2P_{\Delta-}=2P_{\Delta+}\approx\dfrac{\Delta}{\pi}\sqrt{\dfrac{\overline{n_0'^2}}{\overline{n_0^2}}}$ ($P_{\Delta+}$: plus-to-minus zero crossing)

$\Rightarrow P_\Delta=\dfrac{\Delta}{\pi}\sqrt{\dfrac{\overline{n_0'^2}}{\overline{n_0^2}}}=\dfrac{\Delta}{\pi}\sqrt{\dfrac{\frac{1}{3}(2\pi W)^2\cancel{N_0W}}{\cancel{N_0W}}}=\dfrac{2W\Delta}{\sqrt{3}}$ (Prob. of a zero crossing in Δ secs for a random process with an ideal rectangular lowpass spectrum)

Threshold Effect in FM Demodulation

Now, we like to derive an expression for the average number of spikes per second.

Consider sinusoid signal + narrowband Gaussian noise

$$z(t) = A\cos(\omega_0 t) + n(t) = A\cos(\omega_0 t) + n_c(t)\cos(\omega_0 t) - n_s(t)\sin(\omega_0 t)$$
$$= R(t)\cos\left[\omega_0 t + \theta(t)\right],$$

$$R(t) = \sqrt{\left[A + n_c(t)\right]^2 + n_s^2(t)}, \quad \theta(t) = \tan^{-1}\left[\frac{n_s(t)}{A + n_c(t)}\right]$$

Note that the average number of spikes per second is proportional to probability of $R(t)$ crossing the horizontal axis. Fig. D.2(b) shows the when SNR is high, i.e. $R(t)$ does not circle around the origin or $\frac{d\theta}{dt}$ reassembles the sinusoidal message so its integral $= 0$ (see also Fig. 4.30(a) (p. 13 of lec03B)). When SNR is low, $R(t)$ does circle around the origin and the integral of $\frac{d\theta}{dt}$ equals 2π (Fig. D2(c) and Figs. 4.30(b),(c)).

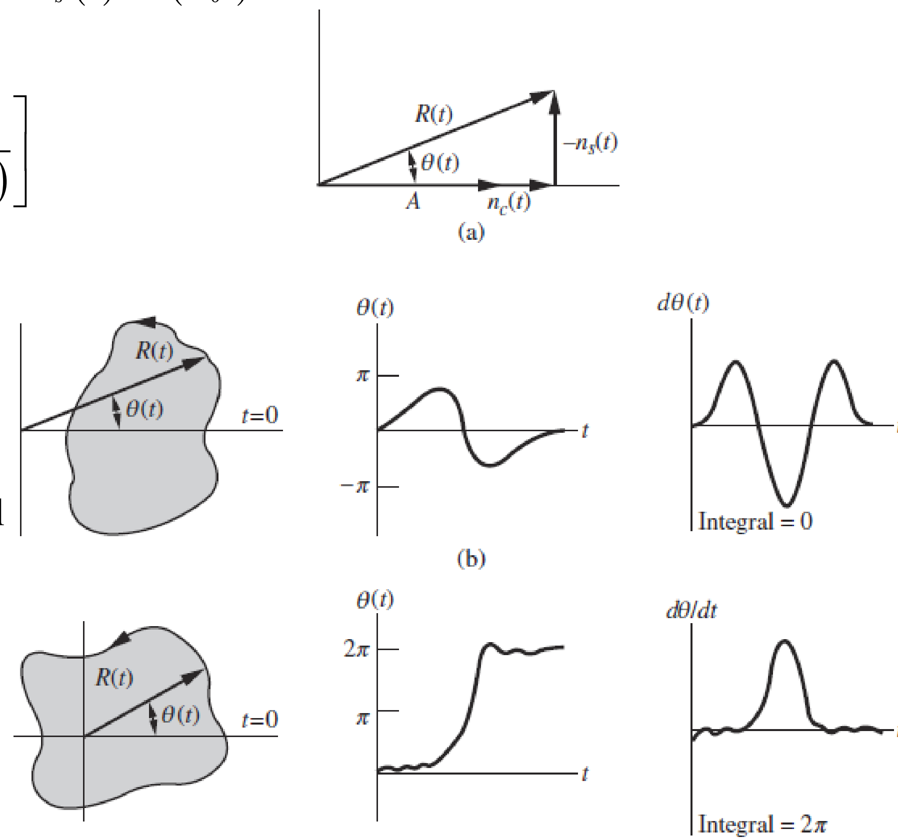
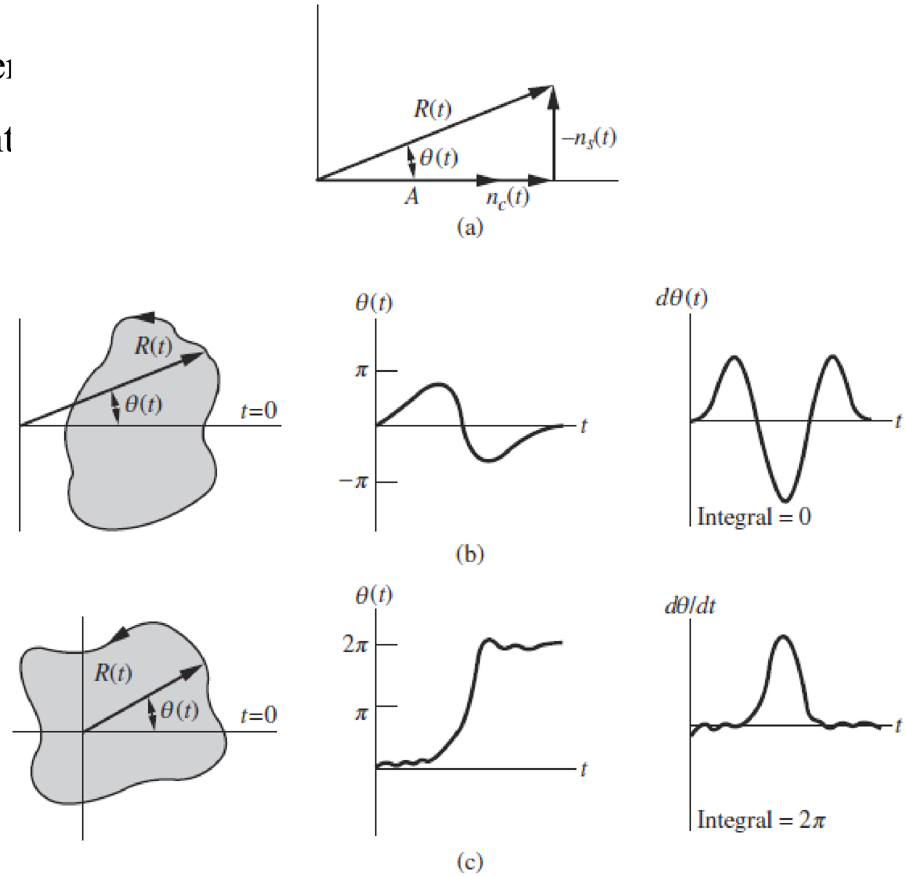


Figure D.2
Phasor diagrams showing possible trajectories for a sinusoid plus Gaussian noise. (a) Phasor representation for a sinusoid plus narrowband noise. (b) Trajectory that does not encircle origin. (c) Trajectory that does encircle origin.

Threshold Effect in FM Demodulation

Assume if $R(t)$ crosses the horizontal axis wher in the 2^{nd} quadrant, then the origin encirclement be completed.



Threshold Effect in FM Demodulation

Consider a small interval Δ seconds in duration, the probability of a counterclockwise encirclement (clockwise is symmetric) $P_{cc\Delta}$ in the interval $(0,\Delta)$ is

$P_{cc\Delta} = P\left[A + n_c(t) \text{ and } n_s(t) \text{ makes } + \text{ to } - \text{ crossing in } (0,\Delta)\right] = P\left[n_c(t) < -A\right] P_{\Delta -}.$

Note that we can replace n with n_c in the expression for $P_{\Delta -}$. Note also that $n(t)$ is a BP process with single-sided BW B, zero-mean Gaussian noise with PSD $N_0 \Rightarrow \overline{n^2(t)} = N_0 B.$

$$\Rightarrow \Pr\left[n_c(t) < -A\right] = \int_{-\infty}^{-A} \frac{e^{-\frac{n_c^2}{2N_0B}}}{\sqrt{2\pi N_0 B}} dn_c = \int_{A/\sqrt{N_0B}}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du = Q\left(\sqrt{\frac{A^2}{N_0 B}}\right).$$

Threshold Effect in FM Demodulation

$$P\left[n_c(t) < -A\right] = Q\left(\sqrt{\frac{A^2}{N_0 B}}\right)$$

Recall from the example that $P_{\Delta-} = \frac{W\Delta}{\sqrt{3}} = \frac{B\Delta}{2\sqrt{3}}$ since $W = \frac{B}{2}$

$$\Rightarrow P_{cc\Delta} = P\left[n_c(t) < -A\right] P_{\Delta-} = \frac{B\Delta}{2\sqrt{3}} Q\left(\sqrt{\frac{A^2}{N_0 B}}\right).$$

Prob. of clockwise encirclement

Due to symmetry, same result will be obtained for $P_{c\Delta}$.

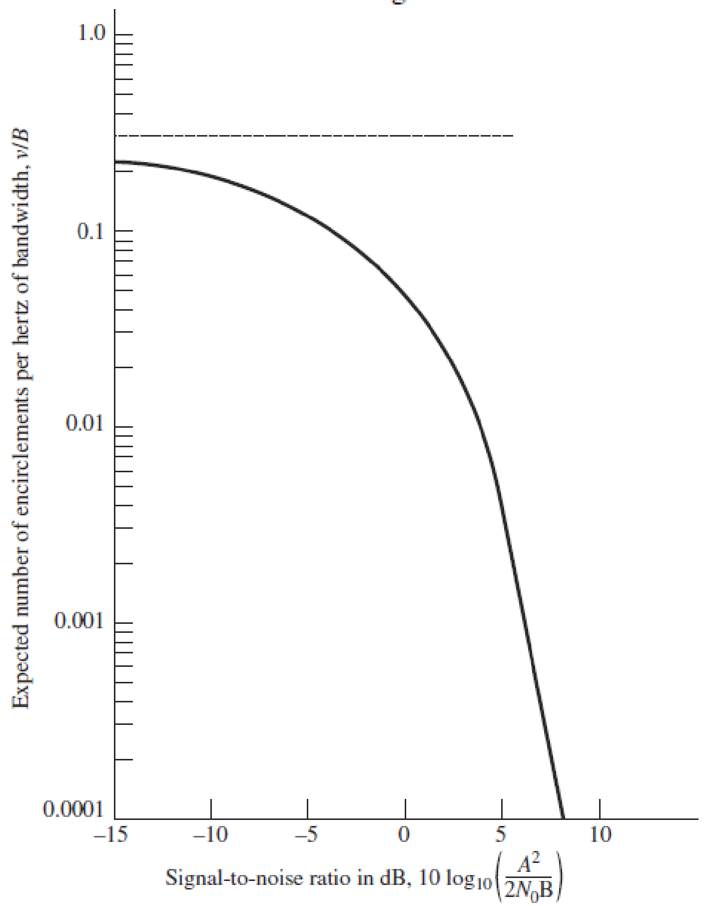
Let $v \triangleq$ expected number of encirclements per second

$$= \frac{1}{\Delta}\left(P_{cc\Delta} + P_{c\Delta}\right) = \frac{B}{\sqrt{3}} Q\left(\sqrt{\frac{A^2}{N_0 B}}\right).$$

Note that $(\mathrm{SNR})_T = \frac{P_T}{N_0 B} = \frac{A^2}{2N_0 B} \;\Rightarrow\; v = \frac{B}{\sqrt{3}} Q\left(\sqrt{2(\mathrm{SNR})_T}\right)$

If $(\mathrm{SNR})_T \ll 1$, $v \approx \frac{B}{\sqrt{3}}\frac{1}{2} = 0.2887B$

Figure D.3
Rate of origin encirclements as a function of signal-to-noise ratio.



Threshold Effect in FM Demodulation

Above results does not provide statistics of the number of impulses, N, in a time interval T. Suppose we model the impulses as periodic so that the noise process can be written as in the form

$$x(t) \triangleq \sum_k n_k \delta(t - kT_s),$$

where $R_n(kT_s) = 0,\ \ \text{for } k = \pm 1, \pm 2, \ldots\ \ \text{(orthogonal)}.$

Recall from Ch. 6:

$$S_x(f) = \lim_{T_s \to \infty} \frac{1}{T_s} E\left[\left|F\left\{x_{T_s}(t)\right\}\right|^2\right] = \frac{R_n(0)}{T_s} = f_s \overline{n_k^2}$$

For FM spikes, $x(t) \triangleq \left.\dfrac{d\theta}{dt}\right|_{\text{impulse}} = \sum_k \pm 2\pi \delta(t - t_k),$

t_k: Poisson point process with average rate v

$$\Rightarrow S_x(f) = v\overline{a_k^2} = v(2\pi)^2 = 4\pi^2 \frac{B}{\sqrt{3}} Q\left(\sqrt{\frac{A^2}{N_0 B}}\right), \quad -\infty < f < \infty$$

Threshold Effect in FM Demodulation

General case: with message

$$z(t) = A\cos\left\{2\pi\left[f_c + \delta f(t)\right]t\right\} + n(t),$$

$\delta f(t)$: frequency deviation.

This frequency deviation causes the spike rate to increase by δv.

It can be shown (approximately) the average change of spike rate is

$$\overline{\delta v} = \overline{\left|\delta f\right|}\exp\left(-\frac{A^2}{2N_0B}\right),$$

$\overline{\left|\delta f\right|}$: average of the magnitude of the frequency deviation

$\Rightarrow$ the total PSD of the impulse (spike) noise is $S_x(f) = \left(v + \overline{dv}\right)(2\pi)^2$

Example: $\delta f(t) = f_d =$ constant > 0

Average noise frequency < $\left(f_c + f_d\right)$ signal frequency

$\Rightarrow$ noise phaser has more clockwise rotations relative to the signal power

$\Rightarrow$ more negative spikes than positive spikes

Can be shown that $\overline{\left|\delta f\right|} = f_d$ in this case.

Threshold Effect in FM Demodulation

Recall the output of the predetection filter:

$$e_1(t) = x_c(t) + n(t) = A_c \cos\left[\omega_c t + \phi(t)\right] + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$
$$= A_c \cos\left[\omega_c t + \phi(t)\right] + r_n(t)\cos\left[\omega_c t + \phi_n(t)\right]$$

$r_n(t)$: Rayleigh distribution

$\phi_n(t)$: uniform distribution

$\overline{n_c^2} = \overline{n_c^2} = N_0 B_T$

Let $\phi_n(t) + \omega_c t = \phi_n(t) - \phi(t) + \omega_c t + \phi(t)$

$\Rightarrow e_1(t) = \cdots = R(t)\cos\left[\omega_c t + \phi(t) + \phi_e(t)\right] = R(t)\cos\left(\omega_c t + \psi(t)\right),$

where $\phi_e(t) = \tan^{-1}\left(\dfrac{r_n(t)\sin\left[\phi_n(t) - \phi(t)\right]}{A_c + r_n(t)\cos\left[\phi_n(t) - \phi(t)\right]}\right)$

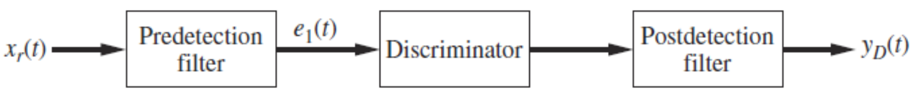


Figure 8.9
Angle demodulation system.

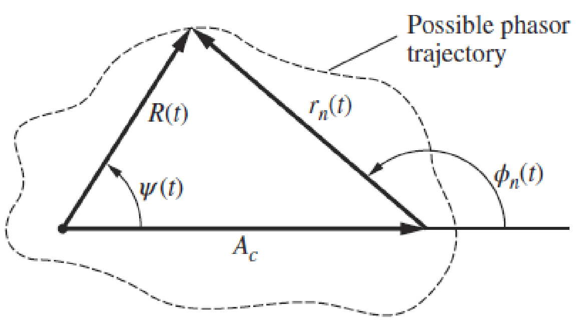


Figure 8.11
Phasor diagram near threshold resulting in spike output (drawn for $\theta = 0$).

- As threshold is approached ➔ $|r_n(t)| > A_c$ for at least part of the time
- $R(t)$ can encircle the origin as $\phi_n(t)$ can be near $-\pi$
- When $R(t)$ is near origin, small change in $\phi_n(t)$ results in rapid change in $\psi(t)$ ➔ discriminator output is large

Threshold Effect in FM Demodulation

FM demodulator output: $y_D(t) = \dfrac{K_D}{2\pi}\dfrac{d\psi}{dt}$.

Since PSD of the spike noise is $S_{\psi'}(f) = \left(v + \overline{dv}\right)(2\pi)^2$

$\therefore$ the spike noise PSD in $y_D(t)$ is

$$S_{D\delta} = K_D^2 v + K_D^2 \overline{dv} = K_D^2 \frac{B_T}{\sqrt{3}} Q\left(\sqrt{\frac{A_c^2}{N_0 B_T}}\right) + K_D^2 \overline{|\delta f|} \exp\left(-\frac{A_c^2}{2N_0 B_T}\right)$$

$\approx$ constant , so it's a white noise

Spike noise power at postdetection filter:

Postdetection filter has $2W$ BW

$$\Rightarrow N_{D\delta} = 2WS_{D\delta} = K_D^2 \frac{2B_T W}{\sqrt{3}} Q\left(\sqrt{\frac{A_c^2}{N_0 B_T}}\right) + K_D^2 2W \overline{|\delta f|} \exp\left(-\frac{A_c^2}{2N_0 B_T}\right)$$

Threshold Effect in FM Demodulation

Recall the addtive Gaussian noise power at the postdetection output: $N_{DF} = \frac{1}{3} K_D^2 W^2 \left(\frac{P_T}{N_0 W} \right)^{-1}$

$\Rightarrow$ Total noise power at postdetection output $=$ additive Gaussian noise power + Spike noise power

$$= N_{DF} + N_{D\delta}$$

$$= \frac{1}{3} K_D^2 W^2 \left(\frac{P_T}{N_0 W} \right)^{-1} + \underbrace{K_D^2 \frac{2B_T W}{\sqrt{3}} Q\left(\sqrt{\frac{A_c^2}{N_0 B_T}} \right) + K_D^2 2W \overline{|\delta f|} \exp\left(-\frac{A_c^2}{2N_0 B_T} \right)}_{\text{spike noise}}$$

- If operating above threshold, terms are negligible as they are $\ll 1$
- Affected by the message signal

Spike noise affected by message signal because

1) $(\mathrm{SNR})_T = \dfrac{A_c^2}{2N_0 B_T}$, where $\dfrac{A_c^2}{2}$ (the 1/2) comes from the message.

2) since the modulation affects $\overline{dv} = \overline{|\delta f|} \exp\left(-\dfrac{A^2}{2N_0 B} \right)$, i.e. average increase of spike noise,

and $\overline{|\delta f|}$ is caused by modulation, hence $(\mathrm{SNR})_D$ is affected by the message through $\overline{|\delta f|}$

Threshold Effect in FM Demodulation

Postdetection output SNR:

$$(\mathrm{SNR})_D = \frac{S_{DF}}{N_{DF} + N_{D\delta}}$$

$$= \frac{K_D^2 f_d^2 \overline{m_n^2}}{\frac{1}{3}K_D^2 W^2 \left(\frac{P_T}{N_0 W}\right)^{-1} + K_D^2 \frac{2B_T W}{\sqrt{3}} Q\left(\sqrt{\frac{A_c^2}{N_0 B_T}}\right) + K_D^2 2W\overline{|\delta f|}\exp\left(-\frac{A_c^2}{2N_0 B_T}\right)}$$

$$= \frac{f_d^2 \overline{m_n^2}}{\frac{1}{3}W^2 \left(\frac{P_T}{N_0 W}\right)^{-1} + \frac{2B_T W}{\sqrt{3}} Q\left(\sqrt{\frac{A_c^2}{N_0 B_T}}\right) + 2W\overline{|\delta f|}\exp\left(-\frac{A_c^2}{2N_0 B_T}\right)} \qquad \left(\text{cancel } K_D^2\right)$$

$$= \frac{3\left(\frac{f_d^2}{W^2}\right)\overline{m_n^2}\frac{P_T}{N_0 W}}{1 + \frac{2}{\sqrt{3}}\frac{B_T}{W}\frac{P_T}{N_0 W} Q\left(\sqrt{\frac{A_c^2}{N_0 B_T}}\right) + 6\frac{\overline{|\delta f|}}{W}\frac{P_T}{N_0 W}\exp\left(-\frac{A_c^2}{2N_0 B_T}\right)}$$

(mult. top and bottom by inverse of 1st term in denom.)

Example 8.4

Let $m_n(t)=\sin(2\pi Wt)$. Compute $(\text{SNR})_D$.

Note that $A_m=1$, and $f_m=W \;\Rightarrow\; \beta=\dfrac{A_m f_d}{f_m}=\dfrac{f_d}{W}\Rightarrow f_d=\beta W$

So the peak frequency deviation $\triangleq A_m f_d=f_d$.

From Carson's rule: $B_T=2(D+1)W=2(\beta+1)W\Rightarrow \dfrac{B_T}{W}=2(\beta+1)$

$$(\text{SNR})_D=\frac{S_{DF}}{N_{DF}+N_{D\delta}}$$

$$=\frac{3\left(\frac{f_d^2}{W^2}\right)\overline{m_n^2}\frac{P_T}{N_0W}}{1+\frac{2}{\sqrt{3}}\frac{B_T}{W}\frac{P_T}{N_0W}Q\left(\sqrt{\frac{A_c^2}{N_0B_T}}\right)+6\frac{\overline{|\delta f|}}{W}\frac{P_T}{N_0W}\exp\left(-\frac{A_c^2}{2N_0B_T}\right)}$$

instantaneous freq deviation: $\delta f(t)=f_d m_n(t)=f_d\sin(2\pi Wt)$

$$\Rightarrow \overline{|\delta f|}=2W\int_0^{1/2W}f_d m_n(t)\,dt=2W\int_0^{1/2W}f_d\sin(2\pi Wt)\,dt=\frac{2}{\pi}f_d=\frac{2}{\pi}\beta W$$

Note: $\overline{m_n^2}=\overline{\sin^2(2\pi Wt)}=\dfrac{1}{2}$

numerator: $S_{DF}=3\left(\dfrac{f_d^2}{W^2}\right)\overline{m_n^2}\dfrac{P_T}{N_0W}=\dfrac{3}{2}\beta^2\dfrac{P_T}{N_0W}$

predetect SNR (denominator): $\dfrac{A_c^2}{2N_0B_T}=\dfrac{A_c^2}{2N_0 2(\beta+1)W}\overset{P_T=\frac{A_c^2}{2N_0W}}{=}\dfrac{1}{2(\beta+1)}\dfrac{P_T}{N_0W}$

$$\Rightarrow(\text{SNR})_D=\frac{\frac{3}{2}\beta^2\frac{P_T}{N_0W}}{1+\frac{4}{\sqrt{3}}(\beta+1)\frac{P_T}{N_0W}Q\left(\sqrt{\frac{1}{(\beta+1)}\frac{P_T}{N_0W}}\right)+\frac{12}{\pi}\beta\frac{P_T}{N_0W}\exp\left(-\frac{1}{2(\beta+1)}\frac{P_T}{N_0W}\right)}$$

Example 8.4

$$(\mathrm{SNR})_D = \frac{\frac{3}{2}\beta^2 \frac{P_T}{N_0 W}}{1 + \frac{4}{\sqrt{3}}(\beta+1)\frac{P_T}{N_0 W} Q\left(\sqrt{\frac{1}{(\beta+1)}\frac{P_T}{N_0 W}}\right) + \frac{12}{\pi}\beta \frac{P_T}{N_0 W}\exp\left(-\frac{1}{2(\beta+1)}\frac{P_T}{N_0 W}\right)}$$

Threshold value defined as the value of denominator of $(\mathrm{SNR})_D = 3$ dB or 2

$$\beta\text{=1};\ \frac{P_T}{N_0 W} = 14 \text{ dB} \ \ (= 25.12)$$

$$\Rightarrow 10\log\left(\frac{3}{2}\beta^2 \frac{P_T}{N_0 W}\Big/ 2\right) = 10\log\frac{3(25.12)}{4} = 12.75$$

$$\Rightarrow \frac{P_T}{N_0 W} = 14 \text{ dB is threshold value for } \beta\text{=1}$$

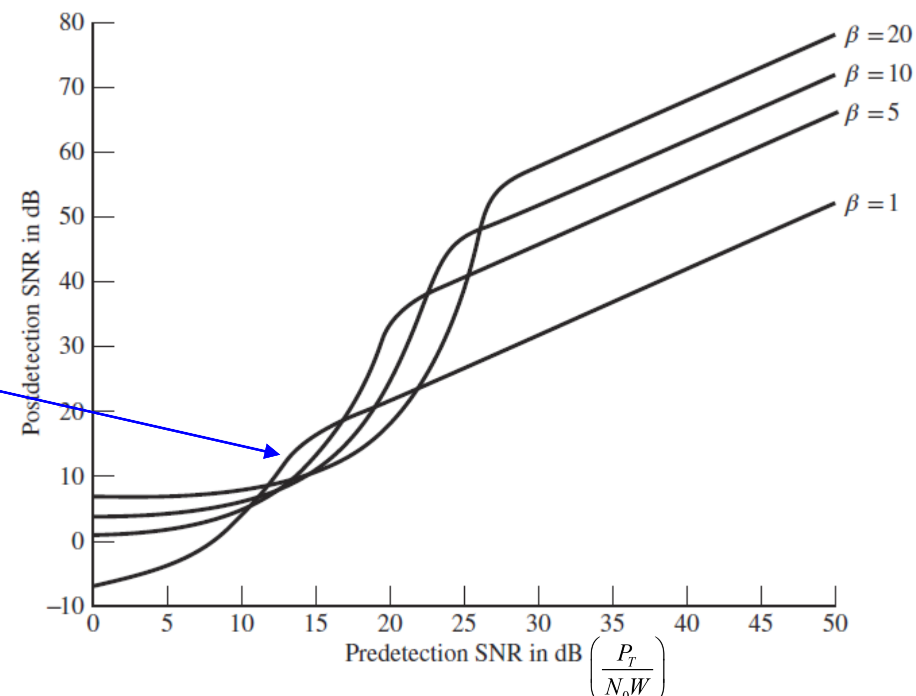


Figure 8.14
Frequency modulation system performance with sinusoidal modulation.

Summary

Table 7.1 Noise Performance Characteristics

System	Postdetection SNR	Transmission bandwidth
Baseband	$\frac{P_T}{N_0 W}$	W
DSB with coherent demodulation	$\frac{P_T}{N_0 W}$	$2W$
SSB with coherent demodulation	$\frac{P_T}{N_0 W}$	W
AM with envelope detection (above threshold) or AM with coherent demodulation. *Note*: E is efficiency	$\frac{EP_T}{N_0 W}$	$2W$
AM with square-law detection	$2\left(\frac{a^2}{2+a^2}\right)^2 \frac{P_T/N_0 W}{1+(N_0 W/P_T)}$	$2W$
PM above threshold	$k_p^2 \overline{m_n^2} \frac{P_T}{N_0 W}$	$2(D+1)W$
FM above threshold (without preemphasis)	$3D^2 \overline{m_n^2} \frac{P_T}{N_0 W}$	$2(D+1)W$
FM above threshold (with preemphasis)	$\left(\frac{f_d}{f_3}\right)^2 \overline{m_n^2} \frac{P_T}{N_0 W}$	$2(D+1)W$

Noise in Pulse-Code Modulation

Total error = quantization error + additive channel error

Sampled and quantized message waveform can be modeled as:

$$m_{\delta q}(t) = \sum_i m(t)\delta(t - iT_s) + \sum_i \varepsilon(t)\delta(t - iT_s)$$

$m(t)$: message

$\varepsilon(t)$: quantization error

- Here we are sampling both as $\varepsilon(t)$ is assumed as a continuous-time function.

  i_{th} sample: $m_{\delta q}(t_i) = m(t_i) + \varepsilon_q(t_i), \quad t_i = iT_s$
- Assume there are q levels in quantizer, with parameters
  - **decision level**: S (width of each level)
  - **quantization/reconstruction level**
- dynamic range: peak-to-peak value of $m(t)$

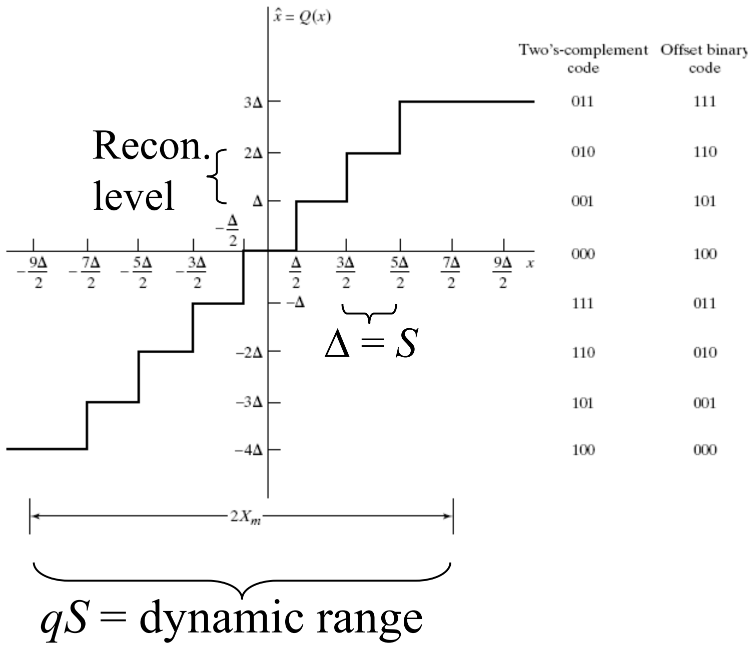


Figure 4.48 Typical quantizer for A/D conversion.

Noise in Pulse-Code Modulation

(1) Quantization error

Assume the quantization error $\varepsilon(t)$ is uniformly distributed $\left(-\frac{1}{2}S, \frac{1}{2}S\right)$

(because the quantization levels are uniformly spaced to be S (called the step-size))

$$\Rightarrow -\frac{1}{2}S \le \varepsilon_q \le \frac{1}{2}S \qquad \text{(note mean = 0)}$$

MSE of $\varepsilon_q(t_i)$:

$$\overline{\varepsilon_q^2} = E\left[\left(\varepsilon_q - \overline{\varepsilon_q}\right)^2\right] = \int_{-S/2}^{S/2} x^2 \frac{1}{S} dx = \frac{S^2}{12}$$

(2) Channel error:

- One PCM sample = n binary pulses (each pulse = 1 symbol or bit, depends on representation) = 1 word
- Bit error probability $= P_b$
- Word error probability = any incorrect bit $= 1-(1-P_b)^n = P_w$

  So $(1-P_b)^n$ means all n bits in a word are correctly received

Noise in Pulse-Code Modulation

Since there are q levels, each with width S, word error is

$$\Rightarrow -\frac{1}{2}qS \le \varepsilon_w \le \frac{1}{2}qS$$

Assume ε_w is uniformly distributed over $\left(-\frac{1}{2}qS, \frac{1}{2}qS\right)$

MSE of ε_w: $\overline{\varepsilon_w^2} = E\left[\left(\varepsilon_w - \overline{\varepsilon_w}\right)^2\right] = \int_{-\frac{1}{2}qS}^{\frac{1}{2}qS} x^2 \frac{1}{qS} dx = \frac{1}{12} q^2 S^2$

Assuming quant. error is indep. of channel error, total PCM noise power is

$$N_D = \overline{\varepsilon_q^2}\left(1 - P_w\right) + \overline{\varepsilon_w^2} P_w$$

first term: contribution of quantization error

second term: contribution from word error

Signal power: assume signal (message) is unif. dist. over $\left(-\frac{1}{2}qS, \frac{1}{2}qS\right)$

$$\Rightarrow S_D = \overline{m^2} = E\left[\left(m - \overline{m}\right)^2\right] = \int_{-\frac{1}{2}qS}^{\frac{1}{2}qS} x^2 \frac{1}{qS} dx = \frac{1}{12} q^2 S^2$$

Noise in Pulse-Code Modulation

$$\Rightarrow (\text{SNR})_D = \frac{S_D}{\overline{\varepsilon_q^2}(1-P_w)+\overline{\varepsilon_w^2}P_w} = \frac{\frac{1}{12}q^2S^2}{\frac{S^2}{12}(1-P_w)+\frac{1}{12}q^2S^2P_w}$$

$$= \frac{1}{q^{-2}(1-P_w)+P_w}$$

If $q = 2^n$ (n bits are used)

$$\Rightarrow (\text{SNR})_D = \frac{1}{2^{-2n}+P_w\left(1-2^{-2n}\right)}$$

- Note that this analysis has been carried using a uniform quantizer
- Normally, the optimal (usu. MSE based) quantizer is designed according to the signal characteristics, i.e. its PDF is taken into account

➔ Lloyd-Max quantizer

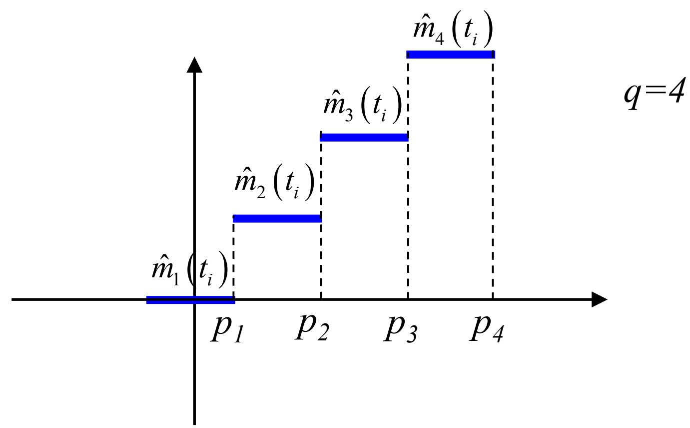
Note that P_w is largely determined by the SNR $\frac{P_T}{N_0W}$ and wordlength n

If P_w small $\quad \Rightarrow \quad (\text{SNR})_D \approx 2^{2n} \quad \Rightarrow 10\log(\text{SNR})_D = 6.02n$

This implies every bit added to represent a word results in ~ 6 dB gain SNR.

Remarks: Quantizer

- Note that this analysis has been carried using a uniform quantizer
- Normally, the optimal (usu. MSE based) quantizer is designed according to the signal characteristics, i.e. its PDF is taken into account
  - Lloyd-Max quantizer
    - Decision and reconstruction level are design parameters that are designed iteratively to minimize the MSE between the actual word and quantized word
    - If optimal decision and reconstruction level are found, then quantizer will be a Lloyd-Max quantizer
      - Converse is not necessarily true
        - Necessary, but not sufficient
  - Proper design minimizes distortion (*D*)
  - Does not take into account of rate (*R*)
  - *D* and *R*: conflicting design objectives



$$\min_{t_i,\hat{m}_k(t_i)} \varepsilon = \sum_{k=1}^{q} \int_{p_k}^{p_{k+1}} \left(m(t_i)-\hat{m}_k(t_i)\right)^2 p_M\left(m(t_i)\right) dm(t_i)$$

Companding

- Analog compression technique that reduces amount of bits/symbol used by increasing the # of bits when amplitude of signal is small and decreasing it when is high
  - Why? Loudness of sound detected by human beings follows a logarithm model
  - No need to uniformly quantized data → waste of bits
- Two solutions:
  - Nonlinear quantization
  - Companding
    - Forces low amplitude signal to run through more quantization levels than high amplitude signal

Companding

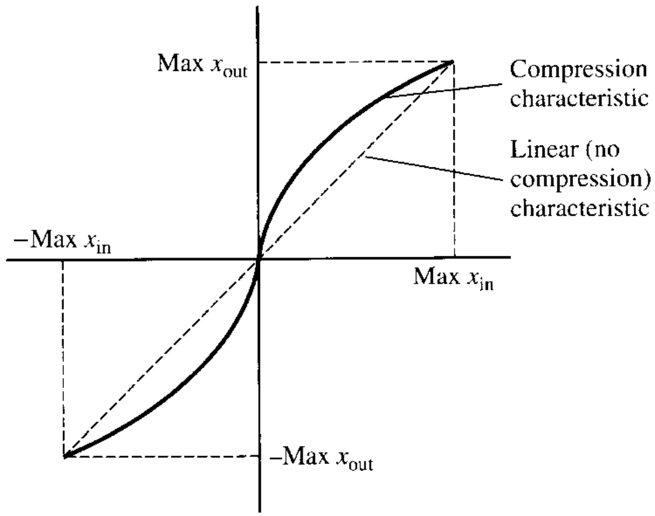


Figure 7.18
Input–output compression characteristic.

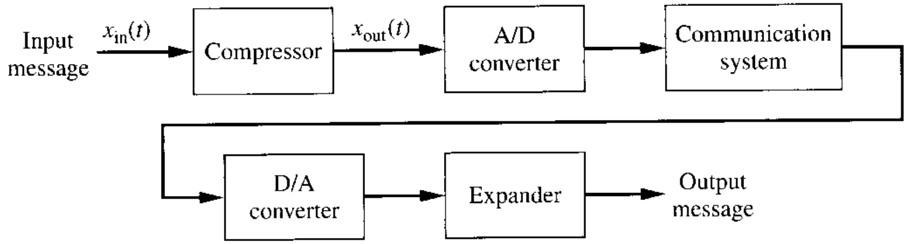


Figure 7.19
Example of companding.

Companding

Recall MSE of $\varepsilon_q(t_i)$: $\overline{\varepsilon_q^2} = E\left[\left(\varepsilon_q - \overline{\varepsilon_q}\right)^2\right] = \int_{-S/2}^{S/2} x^2 \frac{1}{S} dx = \frac{S^2}{12}$

So level of quantization noise is indep. of the signal magnitude since

$\Rightarrow$ SNR from quantization: $(\mathrm{SNR})_Q = \dfrac{\overline{m^2}}{\dfrac{S^2}{12}} = 12\dfrac{\overline{m^2}}{S^2}$

Two solutions:

(1) Non-uniform quantizer: small amplitude steps are small for small amplitudes and larger for large amplitude portions of the signal. Called a Lloyd-Max quantizer

(2) Another solution: Pass analog signal through a nonlinear amplifer before A/D (sampling+quantization+encoding). Then, do the reverse at receiver

$\Rightarrow$ Nonlinear mapping - compressor (transmitter) + expander (receiver)

- change in low-amplitude signal is now forced through more quantization levels than same change in high-amplitude signal $\Rightarrow$ less quantization error

Lloyd-Max Quantizer (Optimal in MSE sense)

Note that

$$\varepsilon = E\left[\left(x[n]-\hat{x}_k[n]\right)^2\right] = \int_{x[n]} \left(x[n]-\hat{x}_k[n]\right)^2 p_X\left(x[n]\right) dx$$

Therefore, the optimal MSE quantizer needs to minimize ε, i.e.

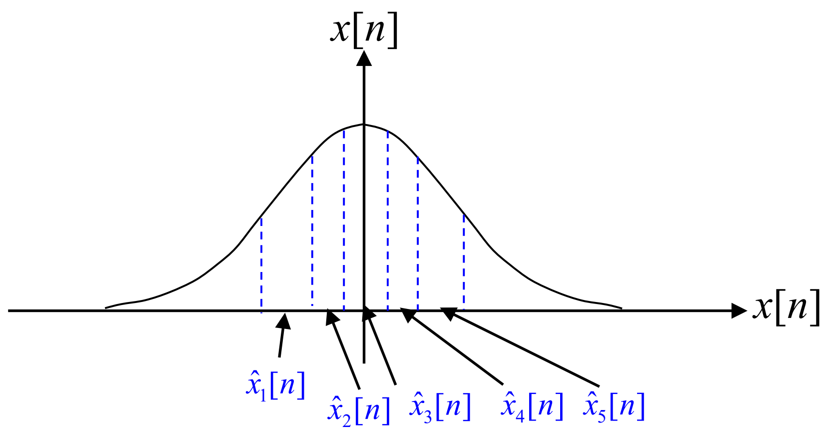
$$\min_{t_k, \hat{x}_k[n]} \varepsilon = \min_{t_k, \hat{x}_k[n]} \int_{x[n]} \left(x[n]-\hat{x}_k[n]\right)^2 p_X\left(x[n]\right) dx$$

Lloyd-Max Quantizer

Decision Level for Lloyd-Max Quantizer (Gaussian Distribution)

Requires an iterative algorithm to solve the problem

- Choose initial value of $\{t_k\}^{(0)}$
- Compute $\{x_k[n]\}^{(0)}$
- Evaluate ε
- Compute $\{t_k\}^{(1)}$
- Compute $\{x_k[n]\}^{(1)}$
- …



5-level Lloyd-Max quantizer for Gaussian-distributed signal $x[n]$

Note: The decision levels are closer together in areas of higher probability (i.e. $p_X(x[n])$ high) so that $\left(x[n]-\hat{x}_k[n]\right)^2$ is small, thus making ε will be smaller

Uniform Quantizer (Special case of Lloyd Max)

Suppose $p_X\left(x[n]\right)=\frac{1}{L}$. Compute $\hat{x}_k\left[n\right]$ using the relationships derived above for the Lloyd-Max quantizer.