Discrete-Time Signals and Systems

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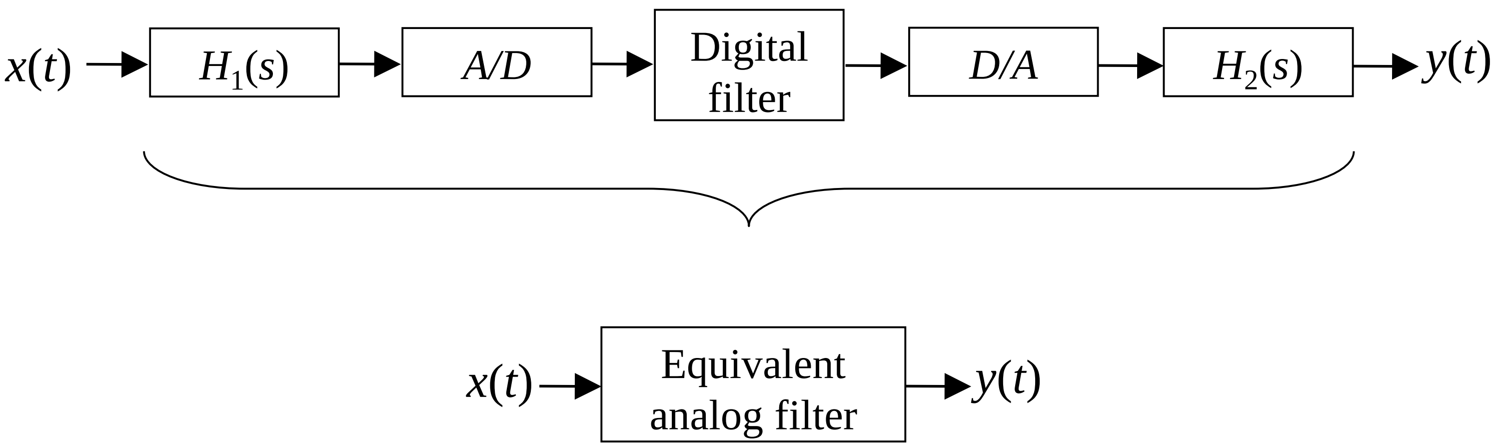
History

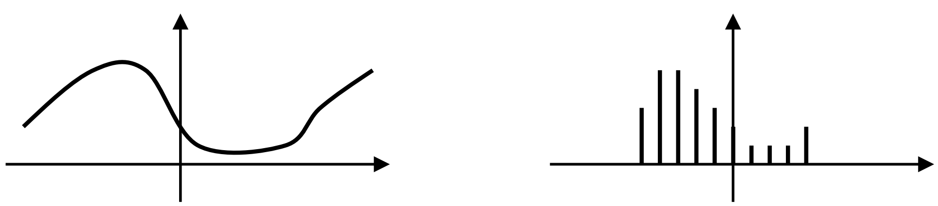
- Before 1950's: analog signals/systems
- 1950's: Digital computer
- 1960's: Fast Fourier Transform (FFT) (turning point)
- 1980's: Real-time VLSI digital signal processor

Typical Digital Signal Processing System



Discrete-Time Signals: Sequences

- Continuous-time signal
  - Defined along a continuum of time: $x(t)$
- Continuous-time system
  - Operates on and produces continuous-time signals
- Discrete-time signal
  - Defined at discrete "times", i.e. $x[n]$ contains a sequence of numbers
  - Anything in between the discrete times is **undefined**
- Discrete-time system
  - Operates on and produces discrete-time signals
- *Digital signals* usually refer to the *quantized* discrete-time signals, i.e. the amplitude is only defined for certain values



Sampling

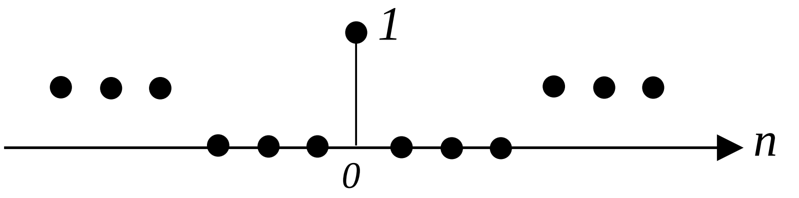
- Very often, $x[n]$ is obtained by sampling $x(t)$
  - $x[n] = x(nT)$, T: sampling period
    - T is often not important in the discrete-time signal analysis

Unit Sample Sequence

- Unit sample sequence (Kronecker delta function)

  $$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

  It is often called the discrete-time impulse or simply impulse (some books call it *unit pulse sequence*)
  - Different from Dirac delta function, i.e. $\delta[0]$ is well-defined.



Unit Step Sequence

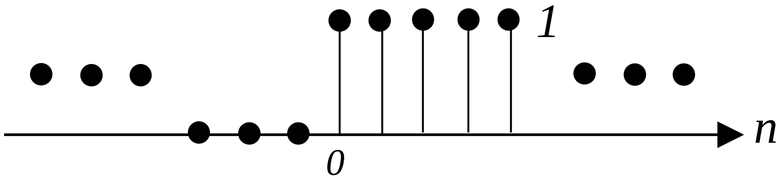
- Unit step sequence

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0 \end{cases}$$

  - Unlike $u(t)$, $u(0)$ is well-defined
  - Also,

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k],$$

$$\delta[n] = u[n] - u[n-1]$$



Exponential Sequence

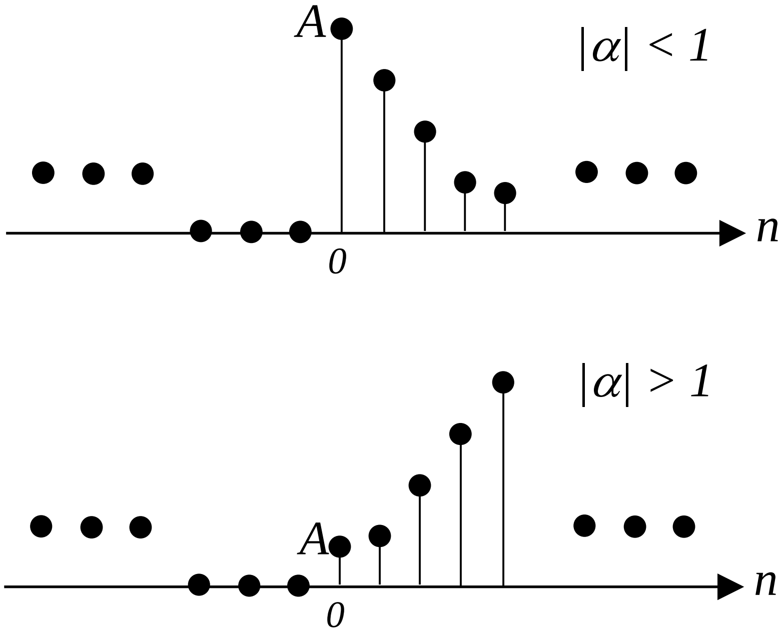
- Exponential sequence

$$x[n] = A\alpha^n$$

  - Combining basic sequences

$$x[n] = \begin{cases} A\alpha^n, & n \geq 0, \\ 0, & n < 0 \end{cases}$$

is equivalent to $x[n] = A\alpha^n u[n]$



Complex Exponential Sequences

magnitude

$$x[n] = A\alpha^n, \text{ where } A = |A|e^{j\phi}, \text{ and } \alpha = |\alpha|e^{j\omega_0}.$$

Hence,

$$x[n] = |A||\alpha|^n e^{j(\omega_0 n + \phi)}$$
$$= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi)$$

Sinusoidal Sequences

- Sinusoidal sequences

$$x[n] = A\cos(\omega_0 n + \phi), \quad \text{for all } n$$

- *A*: amplitude, $\omega_0 = 2\pi f_0$, ϕ: *phase*
- It can be viewed as a sampled continuous-time sinusoidal.
- Condition for being periodic with period *N*, i.e. $x[n] = x[n+N]$, that is,

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 (n + N) + \phi)$$

- Or $\omega_0(n+N) = \omega_0 n + 2\pi k$, where *k*, *n* are integers. *k* is a fixed number while $-\infty < n < \infty$ is a running index

$$\rightarrow \omega_0 N = 2\pi k \rightarrow \omega_0 = \frac{2\pi k}{N}$$

Periodicity of Sinusoidal Sequences

- Consider

$$x_1[n] = \cos(\pi n/4)$$

N = 8 since $x[n+8] = \cos(\pi(n+8)/4) = \cos(\pi n/4+2\pi) = x[n]$

- Increasing the frequency of a DT sequence does not necessarily decrease the period of the signal.

Consider

$$x_2[n] = \cos(3\pi n/8)$$

which has higher frequency than $x_1[n]$. However, it is not periodic with $N = 8$ because $x_2[n+8] = \cos(3\pi(n+8)/8)=\cos(3\pi n/8 + 3\pi) = -x_2[n]$. N=16 is the period

- Some sinusoid sequences are not even periodic. Consider

$$x_3[n] = \cos(n)$$

where $x_3[n+N] \neq x_3[n]$ for all N.

- All these problems are caused by the integer restriction on n, so that the discrete-time sequences are defined only for integer indices n.

Distinguishable Frequencies

- Since

$$\omega_0 = \frac{2\pi k}{N},$$

and

$$\omega_0 \text{ and } \omega_0 + 2\pi r$$

are indistinguishable, therefore,

$$\omega_0 \text{ and } \frac{2\pi k}{N} + 2\pi r$$

are indistinguishable. From this, we see that there will be a total of N distinguishable frequencies for which the corresponding sequences are periodic with period N. One set is when $k = 0, 1, \ldots, N\text{-}1$. This is crucial in understanding complex exponential and sinusoidal sequences, which are also used in discrete-time Fourier analysis (shown later)

Ambiguity of Sinusoidal Sequences

- One discrete-time sinusoid corresponds to multiple continuous-time sinusoids of different frequencies

$$x[n] = A\cos(\omega_0 n + \phi) = A\cos((\omega_0 + 2\pi r)n + \phi), \quad \forall n,$$

where r is any integer

- Typically, we pick up the lowest frequency (r = 0) under the assumption that the original continuous-time sinusoidal has a limited frequency value, $0 \leq \omega_0 \leq 2\pi$

Low and High Frequency

- Consider the analog signal

$$x(t) = A\cos(\Omega_0 t + \phi)$$

As Ω_0 increases, $x(t)$ oscillates more and more rapidly.

- Consider the discrete-time signal

$$x[n] = A\cos(\omega_0 n + \phi)$$

As ω_0 increases from 0 to π, $x[n]$ oscillates more and more rapidly. But as it increases from π *to* 2π, the oscillations become slower.

  - $\omega_0 = (\pi + 2\pi k)$ for any integer k is referred to as high frequency
  - $\omega_0 = 2\pi k$ for any integer k is referred to as low frequency

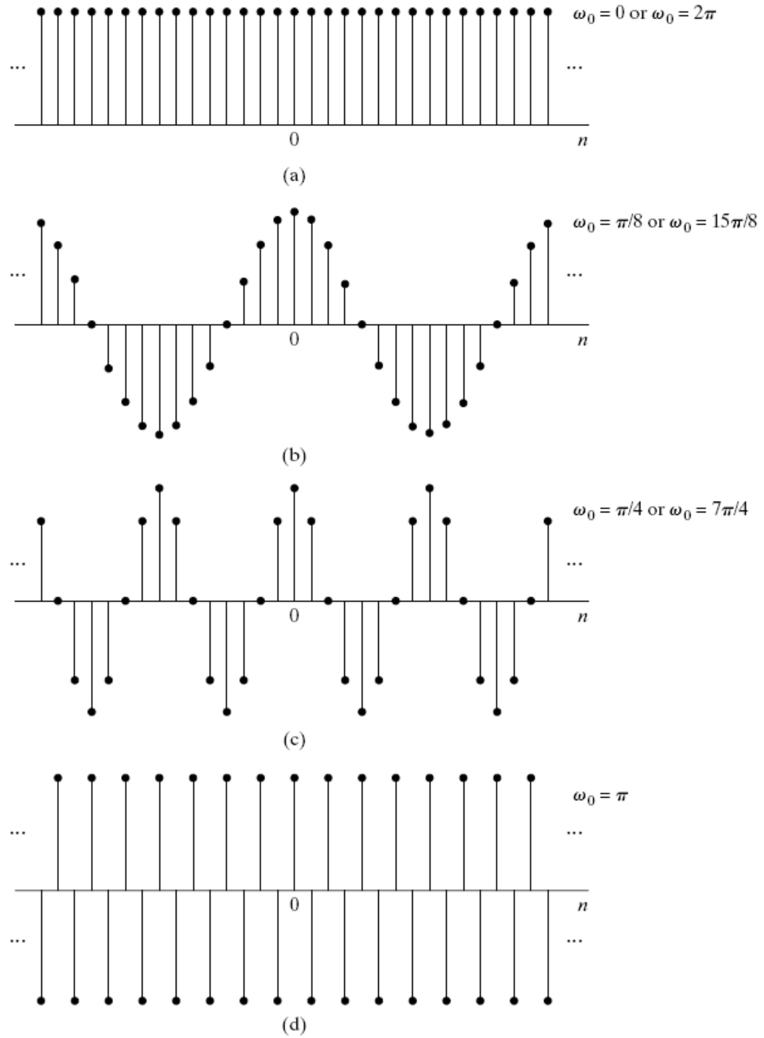


Figure 2.5 $\cos \omega_0 n$ for several different values of ω_0. As ω_0 increases from zero toward π (parts a–d), the sequence oscillates more rapidly. As ω_0 increases from π to 2π (parts d–a), the oscillations become slower.

Discrete Systems

- A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values of $x[n]$ into an output sequence with values $y[n]$

$$y[n] = T\{x[n]\}$$

Ideal Delay

$$y[n] = x[n - n_d], \quad -\infty < n < \infty,$$

where n_d is a fixed positive integer called the delay of the system.

Given the input vector $\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \end{bmatrix}$, $\mathbf{Dx} = \begin{bmatrix} 0 \\ x[0] \\ x[1] \\ \vdots \end{bmatrix}$ is a delayed signal vector,

where $$\mathbf{D} = \begin{bmatrix} \ddots & & & & \\ \ddots & \mathbf{0} & & & \\ & \mathbf{I} & \mathbf{0} & & \\ & & \mathbf{I} & \mathbf{0} & \\ & & & \ddots & \ddots \end{bmatrix}$$

Moving Average

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

$$= \frac{1}{M_1 + M_2 + 1} \left(x[n+M_1] + x[n+M_1-1] + \cdots + x[n] + x[n-1] + \cdots + x[n-M_2] \right)$$

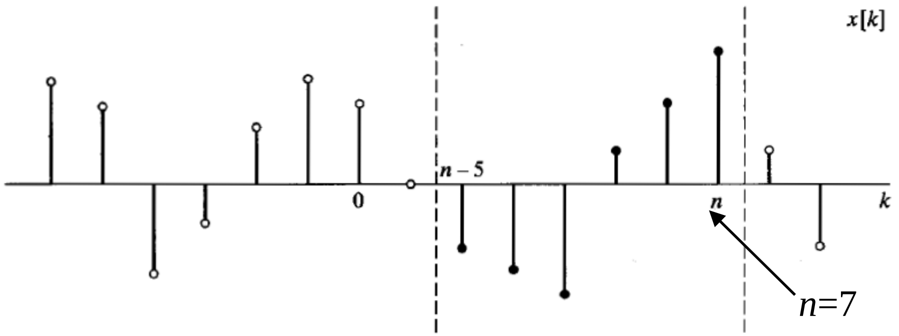


Figure 2.7 Sequence values involved in computing a causal moving average.

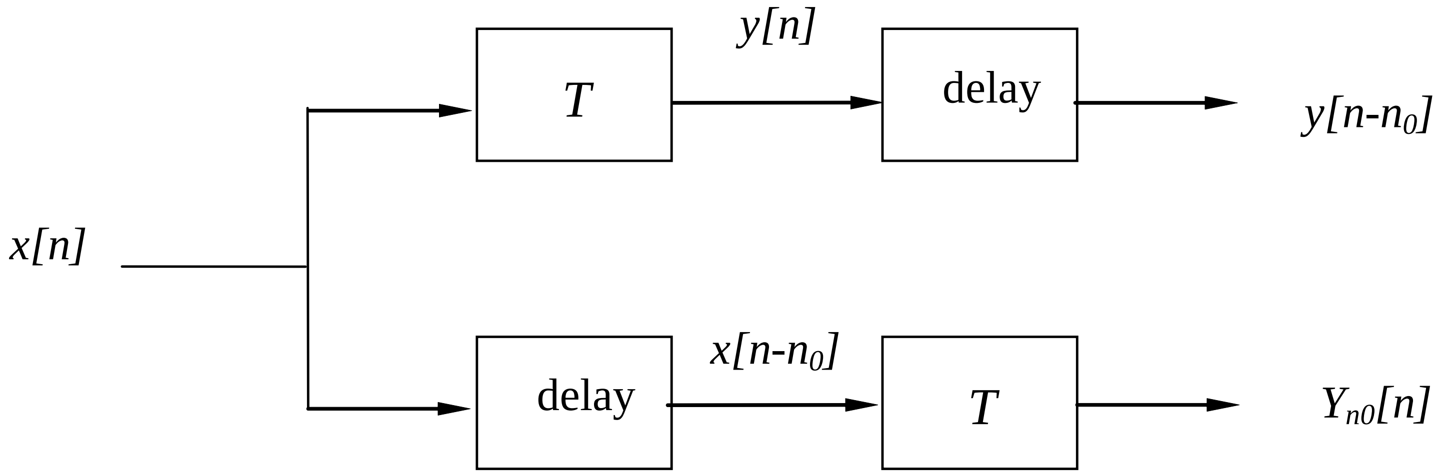
The system computes the nth sample of the output sequence as the average of $(M_1 + M_2 + 1)$ samples of the input sequence around the nth sample. The figure shows an input sequence plotted as a function of a dummy index k and the samples involved in the computation of the output sample $y[n]$ for $n = 7$, $M_1 = 0$, $M_2 = 5$. The output sample $y[7]$ is equal to one-sixth of the sum of all the samples between the vertical dotted lines. To compute $y[8]$ both dotted lines would move one sample to the right

Memoryless, and Linear

- Memoryless
  - If the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n
- Linear: has to satisfy the principle of *superposition*
  - Additivity: $T\{x_1[n]+x_2[n]\} = T\{x_1[n]\}+T\{x_2[n]\}$
  - Scaling: $T\{ax[n]\} = aT\{x[n]\}$

Time/Shift Invariant

- A time shift or delay of the input sequence causes a corresponding shift in the output sequence



Shift-Invariant Example

1) $y[n] = x[\alpha n]$

2) $y[n] = x[n] + c,$ where c is a constant

- Causality
  - For any n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leq n_0$, i.e. output of a causal system does not depend on future values of the input
- BIBO Stability
  - If and only if every bounded input sequence produces a bounded output sequence.

BIBO Stability Examples

1) $y[n] = nx[n]$

Given $|x[n]| < B_x$. Then

$$\left|T\left(x[n]\right)\right| = |nx[n]| \le |n||x[n]| = nB_x$$

As $n \to \infty$, $\left|T\left(x[n]\right)\right|$ will be unbounded, therefore, the system is not BIBO stable

2) $y[n] = x[Mn]$

Given $|x[n]| < B_x$. Then

$$\left|T\left(x[n]\right)\right| = |x[Mn]| < B_x$$

The inequality is true because $x[Mn]$ is equal to $x[n]$, but only retaining every M^{th} sample.

Therefore, the system is BIBO stable

Linear Time-Invariant (LTI) Systems

- A linear time-invariant system is completely characterized by its impulse response.
  - Sequence as a sum of delayed impulses

$$x[n]=\sum_m x[m]\delta[n-m]$$

  - An LTI system due to $\delta[n]$ as an input, i.e.

$$x[n]=\delta[n] \text{ yields } y[n]=h[n] \text{ (impulse response)}$$

  - From the above, we have

$$x[n]=\sum_m x[m]\delta[n-m] \text{ yields } y[n]=\sum_m x[m]h[n-m]$$

- Convolution sum

$$f_3[n]=\sum_m f_1[m]\,f_2[n-m]=f_1[n]*f_2[n]$$

Procedure of Convolution

1) Time reverse: $h[m] \rightarrow h[-m]$
2) Choose as n value
3) Shift $h[-m]$ by n: $h[n-m]$
4) Multiplication: $x[n]\ h[n-m]$
5) Summation over m:

$$y[n] = \sum_{m} x[m]h[n-m]$$

6) Choose another n value, go to Step 3)

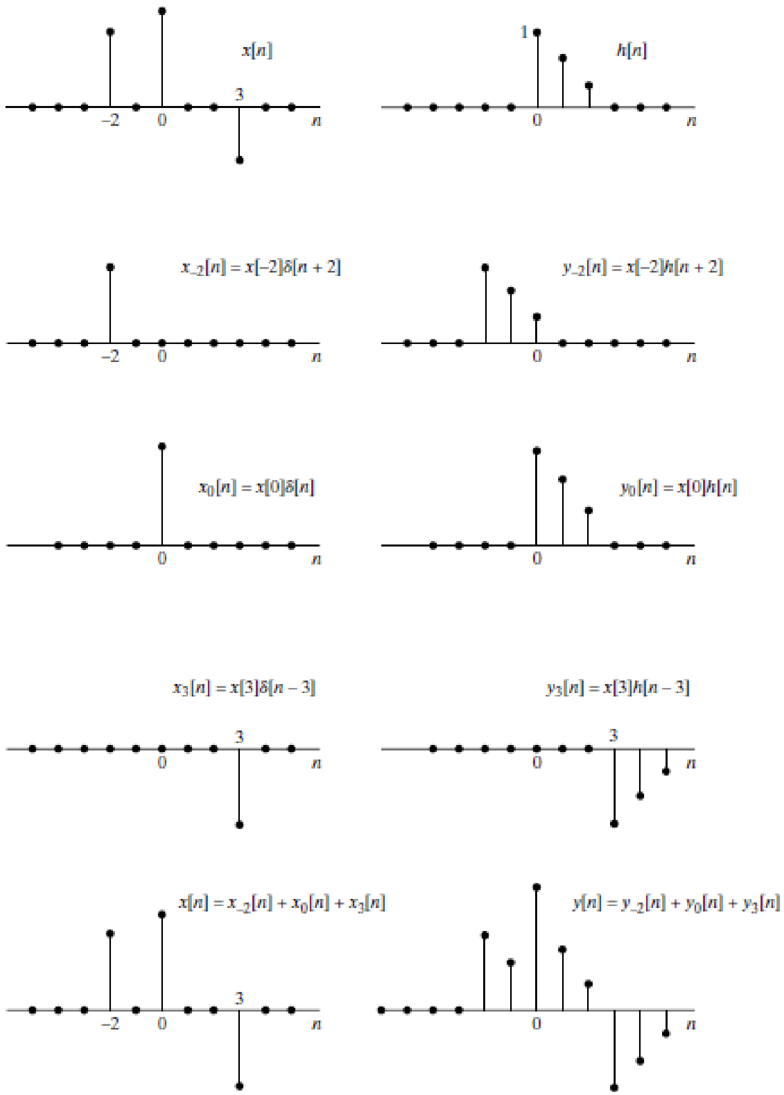


Figure 2.8 Representation of the output of a linear time-invariant system as the superposition of responses to individual samples of the input.

Convolution Example

Determine $y[n]$ for $x[n] = u[-n+2]$ and $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

Convolution Using Matrices and Vectors

Suppose $L_h = 3$ and $L_x = 4$:

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ 0 & h[2] & h[1] & h[0] \\ 0 & 0 & h[2] & h[1] \\ 0 & 0 & 0 & h[2] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} \Leftrightarrow \mathbf{y} = \mathbf{Hx}$$

Suppose we are only interested in a single output sample,

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \mathbf{h}^T \mathbf{x}(n),$$

where

$$\mathbf{h} = \begin{bmatrix} h[0] & h[1] & \cdots & h[L_h - 1] \end{bmatrix}^T$$

$$\mathbf{x}(n) = \begin{bmatrix} x[n] & x[n-1] & \cdots & x[n - L_x + 1] \end{bmatrix}^T$$

Properties of LTI Systems

- Commutative
  - $x[n]*h[n] = h[n]*x[n]$
- Distributive
  - $x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+ x[n]*h_2[n]$
- Cascade connection
  - $h[n] = h_1[n] * h_2[n]$
- Parallel connection
  - $h[n] = h_1[n] + h_2[n]$
- BIBO stability
  - If $h[n]$ is absolutely summable, i.e. $\sum_k |h[k]| = B_h < \infty$
- Causal system
  - $h[n] = 0$, for $n<0$
- Memoryless LTI
  - $h[n] = k\delta[n]$

Frequently Used Systems

Ideal Delay

$$h[n]=\delta[n-n_d] \;\Rightarrow\; y[n]=x[n-n_d]$$

Moving Average

$$h[n]=\begin{cases}\dfrac{1}{M_1+M_2+1}, & -M_1\le n\le M_2\\ 0, & \text{otherwise}\end{cases} \;\Rightarrow\; y[n]=\frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}x[n-k]$$

Accumulator

$$h[n]=u[n] \;\Rightarrow\; y[n]=\sum_{k=-\infty}^{n}x[k]$$

Different Types of LTI Systems

- Finite-duration impulse response (FIR)
  - Its impulse response has only a finite number of non-zero samples
  - Always stable
- Infinite-duration impulse response (IIR)
  - Its impulse response is infinite in duration
- Inverse system
  - System $g[n]$ is the inverse of $h[n]$: $h[n]*g[n] = \delta[n]$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow \boxed{g[n]} \rightarrow y[n]$$

Linear Constant-Coefficient Difference Equation

- Analogous to LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- E.g. first-order system: $y[n] = ay[n-1] + bx[n]$
- General solution: $y[n] = y_p[n] + y_h[n]$
- Homogeneous equation: $\sum_{k=0}^{N} a_k y[n-k] = 0 \quad (x[n] = 0)$

Family of soln: $y_h[n] = \sum_{m=1}^{N} A_m z_m^n$ Why? Substitute $y_h[n]$ into $\sum_{k=0}^{N} a_k y[n-k] = 0$ gives us

$$\sum_{k=0}^{N} a_k \sum_{m=1}^{N} A_m z_m^{n-k} = \sum_{m=1}^{N} A_m z_m^n \sum_{k=0}^{N} a_k z_m^{-k} = 0 \quad \Rightarrow \quad \sum_{k=0}^{N} a_k z_m^{-k} = 0$$

This shows $y_h[n] = \sum_{m=1}^{N} A_m z_m^n$ assume that all N roots of polynomial $\sum_{k=0}^{N} a_k z_m^{-k}$ are distinct.

Have N undetermined coefficients. (Multiple roots is considered in Prob. 2.38)

Linear Constant-Coefficient Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \qquad (*)$$

$$\Leftrightarrow y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \qquad (**)$$

If $x[n],\ y[-1],\ y[-2],\ldots,y[-N]$, are specified, then $y[0]$ can be determined from (**). Then with $y[0], y[-1],\ldots,y[-N+1]$, then $y[1]$ can be found. This is a recursion

To get $y[n]$ for $n<-N$ assuming $y[-1], y[-2],\ldots,y[-N]$ are available, can rearrange (*) into

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k] \qquad (***)$$

so that $y[-N-1], y[-N-2],\ldots$ can be computed recursively

Ex 2.16: Recursively Computing Solution of Diff Eqn

Suppose $y[n] = ay[n-1] + x[n]$, $x[n] = K\delta[n]$, where K is an arbitrary number, and $y[-1] = c$. Then

$$y[0] = ay[-1] + x[0] = ac + K\delta[0] = ac + K$$

$$y[1] = ay[0] + x[1] = a^2c + aK + K\delta[1] = a^2c + aK$$

$$y[2] = ay[1] + x[2] = a^3c + a^2K$$

$$y[3] = ay[2] + x[3] = a^4c + a^3K \qquad \cdots$$

$$\Rightarrow\ y[n] = a^{n+1}c + a^nK, \quad \text{for } n \geq 0$$

Ex 2.16: Recursively Computing Solution of Diff Eqn

To obtain $y[-2], y[-3], \ldots,$ we need to set $N = 1$ in (***). Express the diff. eqn as

$$y[n-1] = -\sum_{k=0}^{0} \frac{a_k}{a_1} y[n-k] + \frac{x[n]}{a_1}. \quad (****)$$

From $y[n] = ay[n-1] + x[n] \Leftrightarrow y[n] - ay[n-1] = x[n] \Rightarrow a_0 = 1, a_1 = -a.$ Then

(****) becomes $y[n-1] = -\frac{1}{-a} y[n] + \frac{1}{-a} x[n] = a^{-1}\left(y[n] - x[n]\right)$

So

$$y[-2] = a^{-1}\left(y[-1] - x[-1]\right) = a^{-1}c \qquad (n = -1)$$

$$y[-3] = a^{-1}\left(y[-2] - x[-2]\right) = a^{-2}c \qquad (n = -2)$$

$$y[-4] = a^{-1}\left(y[-3] - x[-3]\right) = a^{-3}c \qquad (n = -3) \qquad \ldots$$

$$y[n-1] = a^{n}c \quad \Rightarrow \quad y[n] = a^{n+1}c, \quad \text{for } n \le -1$$

Ex 2.16: Recursively Computing Solution of Diff Eqn

$$y[n] = a^{n+1}c + a^n K, \quad \text{for } n \geq 0$$

$$y[n] = a^{n+1}c, \qquad \text{for } n \leq -1$$

$$\Rightarrow y[n] = a^{n+1}c + Ka^n u[n], \quad \forall n$$

- Note we computed solution forward and backward in time, starting with $n = -1$

  $\Rightarrow$ procedure is noncausal

- When $K = 0, x[n] = 0 \;\Rightarrow\; y[n] = a^{n+1}c$ This is the homogeneous solution.
- For system to be linear, output should be zero when input is zero (you can try this yourself). Hence, this system is not linear
- When $x[n] = K\delta[n - n_0], \; y[n] = a^{n+1}c + Ka^{n-n_0}u[n - n_0] \neq y[n - n_0]$

  $\Rightarrow$ system is also not time-invariant

Linear Constant-Coefficient Diff Eqn

- Note that for LCCDE, with additional condition that the system is linear, time-invariant and causal, the solution is unique.
- If the auxiliary conditions (e.g. $y[-1], y[-2]$) are stated as initial-rest condition, i.e. If input $x[n]=0$ for n less than some time n_0, then the output $y[n]$ is constrained to be zero for $n<n_0$. This allows us to use (**) $\sum_{k=1}^{N}\frac{a_k}{a_0}y[n-k]=\sum_{k=0}^{M}\frac{b_k}{a_0}x[n-k]$ to obtain $y[n]$, $n\geq n_0$

Summary of Linear Constant-Coefficient Diff Eqn

- Output for a given input is not unique. Auxiliary conditions are required
- If aux. info is in the form of N sequential values of the output, later values can be obtained by rearranging the diff eqn as a recursive relation running forward in n, and prior values can be obtained by rearranging the diff eqn as a recursive relation running backward in n
- Linearity, time invariance, causality of the system will depend on the aux. conds. If an additional condition is that the system is initially at rest, then the system will be linear, time invariant, and causal

Linear Constant-Coefficient Diff Eqn

Consider Ex. 2.16, with $x[n] = K\delta[n]$, but $y[-1] = 0$ (NOT c anymore)

Old solution: $y[n] = a^{n+1}c + Ka^n u[n], \quad \forall n$

Now:

$y[0] = a \cdot y[0-1] + x[0] = K$

$y[1] = a \cdot y[1-1] + x[1] = Ka + 0 = aK$

$y[2] = ay[2-1] + x[2] = Ka^2 \qquad \cdots$

$\Rightarrow$ New solution: $y[n] = Ka^n u[n]$

This implies also that the impulse response $h[n] = a^n u[n]$ ($h[n] = 0$ for $n < 0$). This is consistent with the causality imposed by the assumption of initial rest

Linear Constant-Coefficient Diff Eqn

If $x[n] = K\delta[n-n_0]$, with $y[-1]=0$, following the same procedure:

$$y[0] = a\cdot y[0-1] + x[0] = 0 \qquad \cdots$$

$$y[n_0] = a\cdot y[n_0-1] + x[n_0] = 0 + K$$

$$y[n_0+1] = a\cdot y[n_0] + x[n_0+1] = aK$$

$$y[n_0+2] = a\cdot y[n_0+1] + x[n_0+2] = a^2K$$

$$\vdots$$

$$\Rightarrow y[n] = Ka^{n-n_0}u[n-n_0]$$

So, this means $y[n_0-1] = \cdots = y[n_0-N] = 0$ if $x[n]=0$ for $n < n_0$

Linear Constant-Coefficient Diff Eqn

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \qquad (*)$$

In Ex. 2.16, we had assumed that $N \geq 1$ in (*). Suppose $N = 0$, no recursion is required to use the difference equation to compute the output, and so no auxiliary conditions are requires. (*) becomes

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k].$$

Let $x[n] = \delta[n]$, then $h[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \delta[n-k] = \begin{cases} \frac{b_n}{a_0}, & 0 \leq n \leq M, \\ 0, & \text{otherwise.} \end{cases}$

This is a causal finite impulse response (FIR) system and its output can be computed nonrecursively

Eigenfunction

Consider

$$x[n] \longrightarrow \boxed{\text{LTI, } h[n]} \longrightarrow y[n]$$

Let $x[n] = e^{j\omega n}$. What is $y[n]$? What is the relation to eigenvalues and eigenvectors?

In general, let $x[n] = e^{j\omega_0 n + \phi}$. What is $y[n]$?

Eigenfunction Example

Let $x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\omega_0 n}e^{j\phi} + \frac{A}{2}e^{-j\omega_0 n}e^{-j\phi}$. Suppose system $h[n]$ is LTI.

$$y_1[n] = \frac{A}{2}e^{j\omega_0 n}e^{j\phi} * h[n], \quad y_2[n] = \frac{A}{2}e^{-j\omega_0 n}e^{-j\phi} * h[n], \quad \Rightarrow y[n] = y_1[n] + y_2[n]$$

Compute total response in terms of amplitude A and $\cos(\cdot)$

Discrete-Time Fourier Transform (DTFT)

- Main idea: Decomposing a signal into its sinusoidal components

  Analysis: $X\left(e^{j\omega}\right)=\sum_{n}x[n]e^{-j\omega n}$

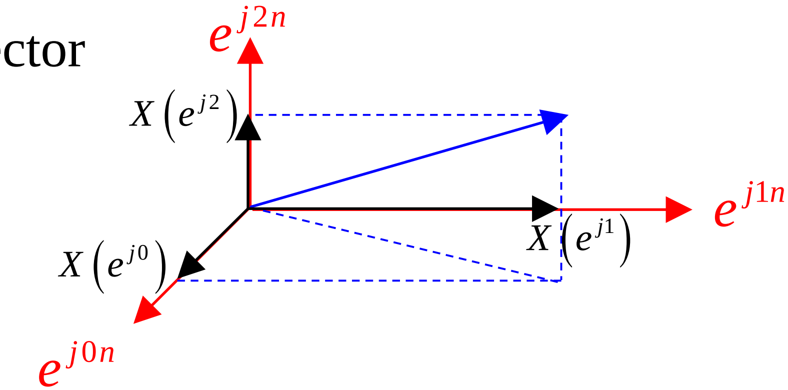
  Synthesis: $x[n]=\frac{1}{2\pi}\int_{\omega=-\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}d\omega$

  - A specific case of projection of vectors.
    - Sinusoidal/exponential functions (of different ω's) form the basis vectors.
    - Signal to be decomposed is the vector

Recall the projection matrix: $\mathbf{P}=\frac{\mathbf{v}\mathbf{v}^H}{\mathbf{v}^H\mathbf{v}}$.

If $\mathbf{v}$ is orthonormal(rewrite as $\mathbf{q}_i$), then

$$\mathbf{a}=\underbrace{\mathbf{q}_1\underbrace{\mathbf{q}_1^H\mathbf{a}}_{\text{analysis}}}_{\text{synthesis}}+\underbrace{\mathbf{q}_2\underbrace{\mathbf{q}_2^H\mathbf{a}}_{\text{analysis}}}_{\text{synthesis}}+\underbrace{\mathbf{q}_3\underbrace{\mathbf{q}_3^H\mathbf{a}}_{\text{analysis}}}_{\text{synthesis}}$$



DTFT

- Fourier transform is also called *Fourier spectrum*
- Magnitude spectrum: $|X(e^{j\omega})|$
- Phase spectrum: $\angle X(e^{j\omega})$
- $X(e^{j\omega})$ is continuous in ω
- $X(e^{j\omega})$ is periodic with period 2π

DTFT Example

Compute the Fourier transform of $x[n] = a^n u[n]$

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \quad \text{if } \left|ae^{-j\omega}\right| < 1 \text{ or } |a| < 1.$$

The last equality is obtained because for the series to converge, i.e.

$\sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n \le \left|\sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n\right| \le \sum_{n=0}^{\infty} \left|ae^{-j\omega}\right|^n < \infty,$ therefore, $\left|ae^{-j\omega}\right| < 1$

or $|a| < \left|e^{j\omega}\right| = 1$ is sufficient.

Convergence of the DTFT

- Not all sequences have DTFT. If the sequence satisfy the following 2 conditions, then the sequence will have a DTFT

1) Absolutely summable sequence

$$\sum_n \left|x[n]\right| < \infty \qquad \text{(uniform convergence)}$$

It is clear from the example in the last slide that as long as $|a|<1$, then $x[n] = a^n u[n]$ is absolutely summable. In fact,

$$\begin{aligned}\left|X\left(e^{j\omega}\right)\right| &= \left|\sum_n x[n] e^{-j\omega n}\right| \\ &\leq \sum_n \left|x[n]\right| \left|e^{-j\omega n}\right| \\ &= \sum_n \left|x[n]\right| \qquad \left(\text{since } \left|e^{-j\omega n}\right| = 1\right) \\ &< \infty\end{aligned}$$

Therefore, $\sum_n \left|x[n]\right| < \infty$ is sufficient condition to guarantee that the Fourier transform exists (converges)

Convergence of DTFT

- All finite sequences are absolutely summable
- Absolute summability guarantees uniform convergence of the Fourier transform
- If a sequence is not absolutely summable, the Fourier transform can still be written under a more relaxed condition of *mean-square convergence*

Convergence of DTFT

2) If the sequence is square-summable, i.e.

$$\sum_n |x[n]|^2 < \infty,$$

then the sequence will have mean-square convergence. Let

$$X\left(e^{j\omega}\right) = \sum_n x[n] e^{-j\omega n},$$

$$X_M\left(e^{j\omega}\right) = \sum_{n=-M}^{M} x[n] e^{-j\omega n}$$

The mean-square convergence of the corresponding sequence, $x[n]$, can then be written as

$$\lim_{M\to\infty} \int_{\omega=-\pi}^{\pi} \left| X\left(e^{j\omega}\right) - X_M\left(e^{j\omega}\right)\right|^2 d\omega = 0$$

Example of Mean-Square Convergence

Let the frequency response of an ideal lowpass filter be

$$H_{\ell p}\left(e^{j\omega}\right)=\begin{cases}1, & |\omega|<\omega_c,\\ 0, & \omega_c<|\omega|\le\pi,\end{cases}$$

then the impulse response is

$$\begin{aligned}h_{\ell p}[n]&=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}e^{j\omega n}d\omega\\ &=\frac{1}{2\pi jn}e^{j\omega n}\Big|_{-\omega_c}^{\omega_c}\\ &=\frac{1}{2\pi jn}\left(e^{j\omega_c n}-e^{-j\omega_c n}\right)\\ &=\frac{\sin(\omega_c n)}{\pi n},\qquad -\infty<n<\infty.\end{aligned}$$

Note that $h_{\ell p}[n]$ is not absolutely summable because the sequence approaches 0 as $n \to 0$, but only at a rate of $\frac{1}{n}$. Therefore, $\sum_n \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$ does not converge uniformly for all ω's. However, evaluating $H_M(e^{j\omega})$, where it is equal to

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{\sin\left[(2M+1)(\omega-\theta)/2\right]}{\sin\left[(\omega-\theta)\right]/2} d\theta,$$

shows that $H_M(e^{j\omega})$ does not approach $H(e^{j\omega})$ due to the oscillation around ω_c.

Therefore, it is indeed true that $\sum_n \frac{\sin(\omega_c n)}{\pi n} e^{-j\omega n}$ does not converge uniformly.

However, $$\lim_{M\to\infty} \int_{\omega=-\pi}^{\pi} \left| H(e^{j\omega}) - H_M(e^{j\omega}) \right|^2 d\omega = 0.$$

Thus, $h_{\ell p}[n]$ is square summable.

Gibbs Phenomenon

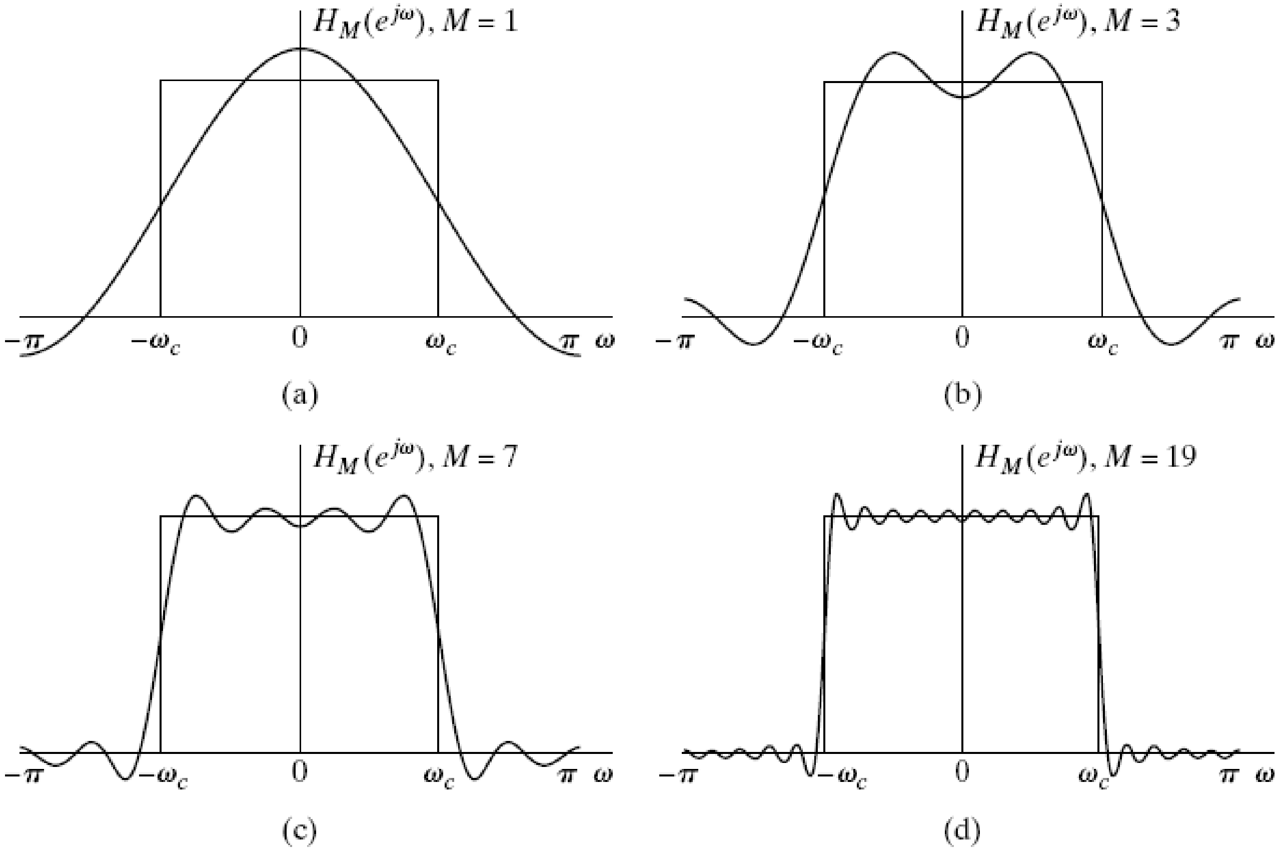


Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

DTFT of Special Functions

Impulse:

$$\delta[n] \leftrightarrow 1$$

$$\delta[n-n_0] \leftrightarrow e^{-j\omega n_0}$$

Constant:

$$1 \leftrightarrow \sum_r 2\pi\delta(\omega + 2\pi r)$$ (This is a periodic impulse train using Dirac delta function)

Note: This sequence is neither absolutely nor square summable. However, it is possible and useful to define the Fourier transform of the sequence to be the periodic impulse train. The impulses here are functions of a continuous variable and therefore are of "infinite height, zero width, and unit area", consistent with the fact that $\sum_r 2\pi\delta(\omega + 2\pi r)$ does not converge. The Fourier transform is justified in principle because substitution of $\sum_r 2\pi\delta(\omega + 2\pi r)$ into the synthesis equation leads to $x[n] = 1$.

DTFT of Special Functions

Complex exponential:

$$e^{j\omega_0 n} \leftrightarrow \sum_r 2\pi\delta\left(\omega - \omega_0 - 2\pi r\right)$$

Note: This is a more general example of the above. Assumes that $-\pi < \omega_0 < \pi$. Using the synthesis equation of the DTFT, we can write

$$x\left[n\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta\left(\omega - \omega_0\right)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}2\pi e^{j\omega_0 n} = e^{j\omega_0 n}$$

For $\omega_0 = 0$, this reduces back to $x\left[n\right] = 1$ (the example above).

Cosine sequence:

$$\cos\left(\omega_0 n + \theta\right) \leftrightarrow \sum_k \pi\left[e^{j\theta}\delta\left(\omega - \omega_0 + 2\pi k\right) + e^{-j\theta}\delta\left(\omega + \omega_0 + 2\pi k\right)\right]$$

Unit step:

$$u\left[n\right] \leftrightarrow \frac{1}{1 - e^{-j\omega}} + \pi\sum_r \delta\left(\omega + 2\pi r\right)$$

DTFT of Special Functions

In summary, when sequences $x[n]$ such as $e^{j\omega_0 n}$ is not absolutely sumable nor square summable, and $\left|X\left(e^{j\omega}\right)\right|$ is not finite for all ω, the statement

$$\sum_n e^{j\omega_0 n} e^{j\omega n} = \delta\left(\omega - \omega_0 + 2\pi k\right)$$

must be interpreted in a special way using generalized functions. Using this, we can extend the concept of a Fourier transform representation to the class of sequences that can be expressed as a sum of discrete frequency components, such as

$$x[n] = \sum_k a_k e^{j\omega_k n}, \quad \text{for } -\infty < n < \infty.$$

Then

$$X\left(e^{j\omega}\right) = \sum_r \sum_k 2\pi\delta\left(\omega - \omega_k - 2\pi r\right)$$

is a consistent Fourier transform representation of $x[n] = \sum_k a_k e^{j\omega_k n}$.

Symmetry Properties of the DTFT

- Any (complex) $x[n]$ can be decomposed into

$$x[n] = x_e[n] + x_o[n]$$

where

$x_e[n] = (x[n] + x^*[-n])/2$ is the *conjugate symmetric part*

$x_o[n] = (x[n] - x^*[-n])/2$ is the *conjugate antisymmetric part*

Remark: $x[n]$ is *conjugate symmetric* if $x[n] = x^*[-n]$

$x[n]$ is *conjugate antisymmetric* if $x[n] = -x^*[-n]$

- On the other hand,

$$X(e^{j\omega}) = \mathrm{Re}\{X(e^{j\omega})\} + j\mathrm{Im}\{X(e^{j\omega})\}$$

Symmetry Properties of the DTFT

- $x_e[n] \Leftrightarrow \mathrm{Re}\{X(e^{j\omega})\}$, $x_o[n] \Leftrightarrow j\mathrm{Im}\{X(e^{j\omega})\}$

  Similarly, $X(e^{j\omega})$ can be decomposed into

  $$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

  where $X_e(e^{j\omega})$ is the *conjugate symmetric part* and

  $X_o(e^{j\omega})$ is the *conjugate antisymmetric part*

- $\mathrm{Re}\{x[n]\} \Leftrightarrow X_e(e^{j\omega})$, $j\mathrm{Im}\{x[n]\} \Leftrightarrow X_o(e^{j\omega})$

  Special case 1: If $x[n]$ is real, then $X(e^{j\omega})$ is conjugate symmetric (magnitude is even, phase is odd)

  Special case 2: If $x[n]$ is conjugate symmetric, then $X(e^{j\omega})$ is real.

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

$$x[n] = a^n u[n] \Leftrightarrow X\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}, \ \text{if } |a| < 1$$

Property 7: $X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$

Property 8: $X_R\left(e^{j\omega}\right) = \dfrac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R\left(e^{-j\omega}\right)$

Property 9: $X_I\left(e^{j\omega}\right) = \dfrac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I\left(e^{-j\omega}\right)$

Property 10: $\left|X\left(e^{j\omega}\right)\right| = \dfrac{1}{\left(1 + a^2 - 2a\cos\omega\right)^{1/2}} = \left|X\left(e^{-j\omega}\right)\right|$

Property 11: $\angle X\left(e^{j\omega}\right) = \tan^{-1}\left(\dfrac{-a\sin\omega}{1 - a\cos\omega}\right) = -\angle X\left(e^{-j\omega}\right)$

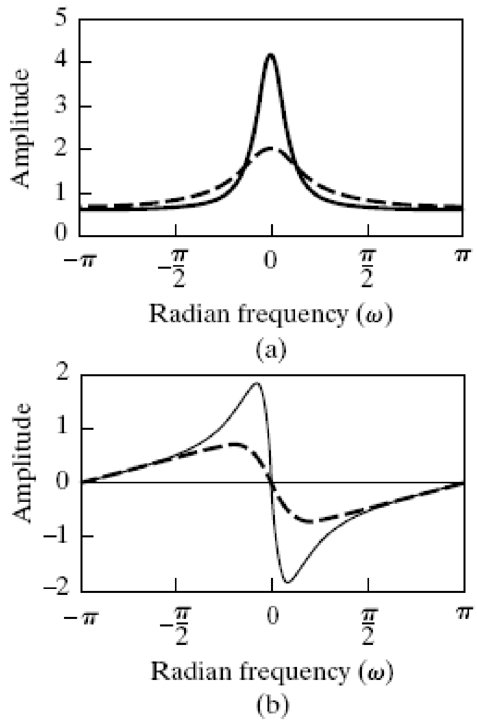


Figure 2.22 Frequency response for a system with impulse response $h[n] = a^n u[n]$. (a) Real part. $a > 0$; $a = 0.9$ (solid curve) and $a = 0.5$ (dashed curve). (b) Imaginary part.

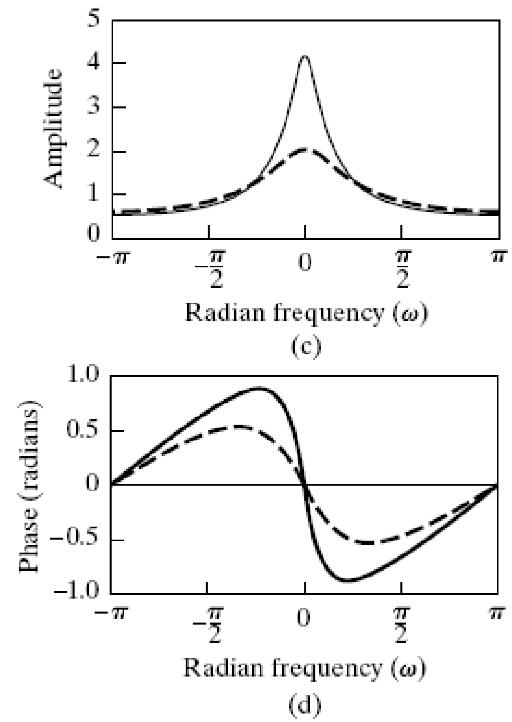


Figure 2.22 (*Continued*) (c) Magnitude. $a > 0$; $a = 0.9$ (solid curve) and $a = 0.5$ (dashed curve). (d) Phase.

Fourier Transform Theorems

Linearity:

If $x[n] \leftrightarrow X(e^{j\omega})$ and $y[n] \leftrightarrow Y(e^{j\omega})$

then $ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$

Time Shift:

If $x[n] \leftrightarrow X(e^{j\omega})$

then $x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$

Frequency Modulation:

If $x[n] \leftrightarrow X(e^{j\omega})$

then $e^{-j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$

Time Reversal:

If $x[n] \leftrightarrow X(e^{j\omega})$

then $x[-n] \leftrightarrow X(e^{-j\omega})$

Differentiation in frequency:

If $x[n] \leftrightarrow X(e^{j\omega})$

then $nx[n] \leftrightarrow j \dfrac{dX(e^{j\omega})}{d\omega}$

Fourier Transform Theorem

Convolution:

$$\text{If } x[n] \leftrightarrow X\left(e^{j\omega}\right) \text{ and } h[n] \leftrightarrow H\left(e^{j\omega}\right)$$

$$\text{then } x[n]*h[n] \leftrightarrow X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

Multiplication:

$$\text{If } x[n] \leftrightarrow X\left(e^{j\omega}\right) \text{ and } w[n] \leftrightarrow W\left(e^{j\omega}\right)$$

$$\text{then } x[n]w[n] \leftrightarrow \frac{1}{2\pi}\int_{\theta=-\pi}^{\pi} X\left(e^{j\omega}\right)W\left(e^{j(\omega-\theta)}\right)d\theta$$

Parseval's Theorem:

$$\text{If } x[n] \leftrightarrow X\left(e^{j\omega}\right)$$

$$\text{then } E = \sum_{n}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

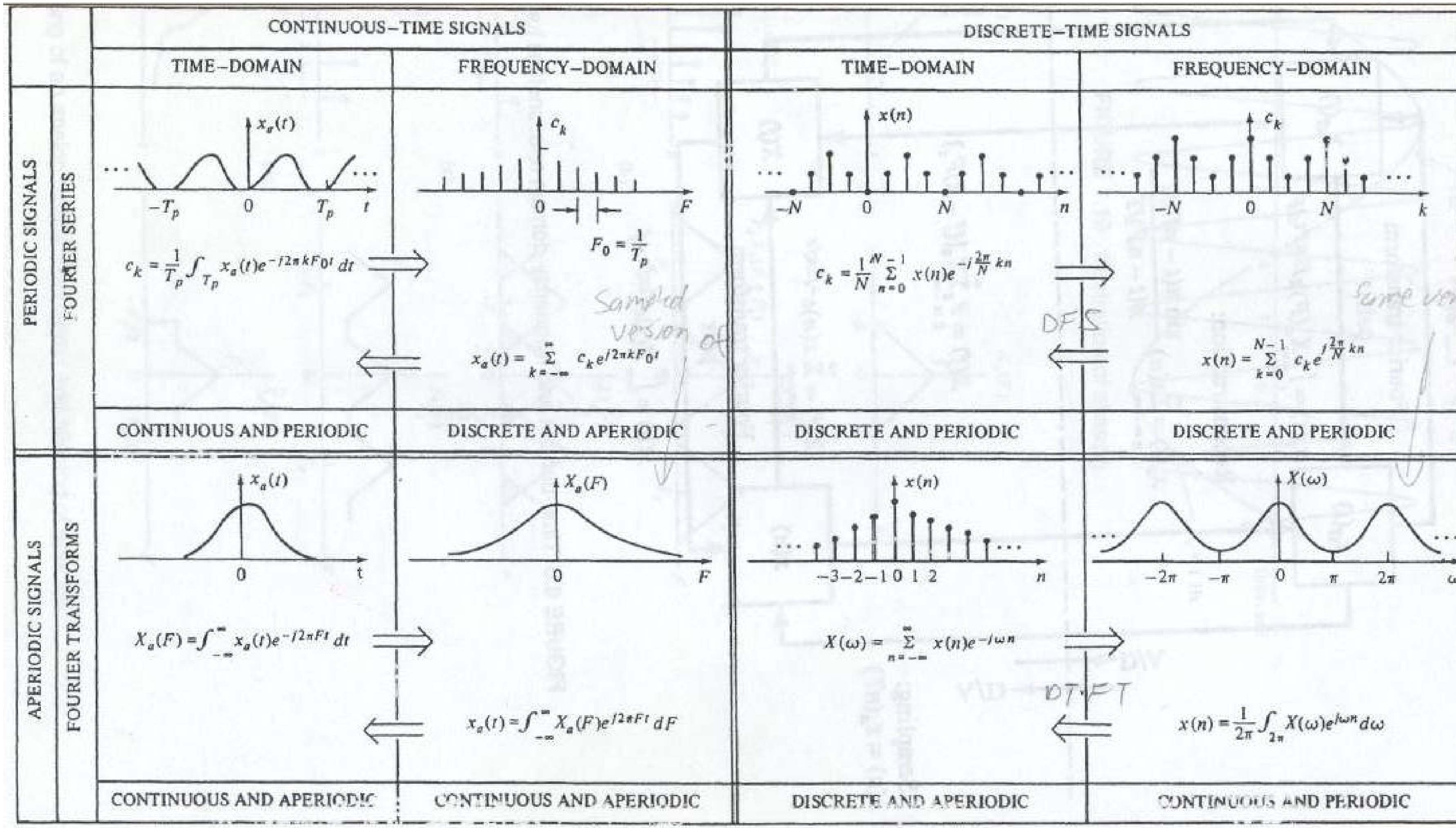


FIGURE 3.31 Summary of analysis and synthesis formulas.