The *z*-Transform

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Movations

- Generalization of the DTFT
  - Some sequences that do not converge for DTFT have valid z-transforms
- Better notation compared to FT in analytical problems (complex variable theory)
- Solving difference equation → algebraic equation
  - similar to using Laplace transform in solving differential equation

Eigenfunction

Consider

$$x[n] \longrightarrow \boxed{\text{LTI, } h[n]} \longrightarrow y[n]$$

Let $x[n] = z_0^n$. What is $y[n]$?

z-transform

Two-sided *z*-Transform (bilateral *z*-Transform)

Forward: $Z\left(x[n]\right) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \equiv X(z)$

From DTFT viewpoint: $Z\left\{x[n]\right\} = F\left\{r^{-n} x[n]\right\}\Big|_{re^{j\omega}=z}$

Inverse: $x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \equiv Z^{-1}\left[X(z)\right]$

The integration is evaluated along a counterclockwise circle on the complex *z* plane with a radius *r*.

Remark: $X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

The DTFT can be viewed as a special case: $z = e^{j\omega}$

Single-sided *z*-Transform (unilateral) – for causal sequenes

Forward: $X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

Interpretation of the z-Transform

- If z is restricted to have unity magnitude, i.e. $|z| = 1$, the z-Transform corresponds to the Fourier transform. That is we can express z in polar form as $z = e^{j\omega}$
  - The z-transform can be written as

$$X\left(re^{j\omega}\right) = \sum_n x[n]\left(re^{j\omega}\right)^{-n}$$
$$= \sum_n \left(x[n]r^{-n}\right)e^{-j\omega n}.$$

  z-transform can now be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence r^{-n}. For $r = 1$, this reduces back to the Fourier transform of $x[n]$.

z-Plane

- ω is the angle between the vector to a point z on the unit circle and the real axis of the complex z-plane
  - Evaluate $z = 1$ corresponds to evaluating the Fourier transform at $\omega = 0$
  - Evaluate $z = j$ corresponds to evaluating the Fourier transform at $\omega = \pi/2$
  - Evaluate $z = -1$ corresponds to evaluating the Fourier transform at $\omega = \pi$
- Continue evaluation of the Fourier transform from $\omega = \pi$ to $\omega = 2\pi$ is equivalent to evaluation from $\omega = -\pi$ to $\omega = 0$

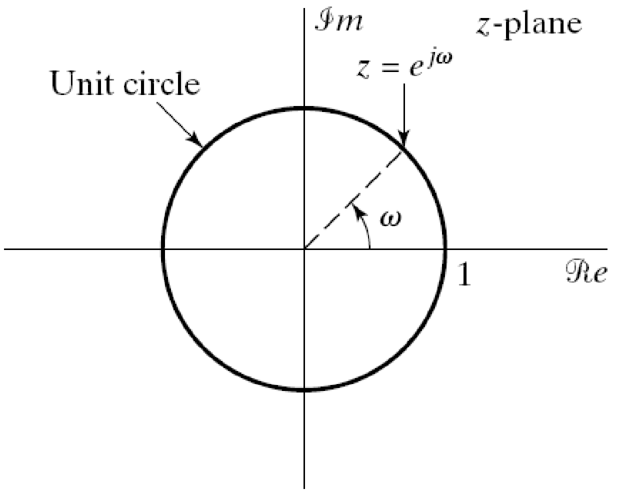

Figure 3.1 The unit circle in the complex z-plane.

Region of Convergence (ROC)

- The ROC is the set of values of *z* for which the z-transform converges
- Uniform convergence
  - If $z=re^{j\omega}$ (polar form), the *z*-Transform converges uniformly if $x[n]r^{-n}$ is *absolutely summable*; that is

$$
\begin{aligned}
\left|X(z)\right| &= \left|X\left(re^{j\omega}\right)\right| \\
&= \left|\sum_n x[n]r^{-n}e^{-j\omega n}\right| \\
&\le \sum_n \left|x[n]r^{-n}\right|\left|e^{-j\omega n}\right| \\
&= \sum_{n=-\infty}^{\infty} |\, x[n]r^{-n}\, | < \infty
\end{aligned}
$$

  - The exponential weight helps with the convergence
  - E.g. $u[n]$ is not absolutely summable, so FT does not converge absolutely. But $r^{-n}u[n]$ is absolutely summable if $r > 1$
  - This implies the *z*-Transform for $u[n]$ exists with a ROC of $|z| > 1$

Region of Convergence (ROC)

- Since convergence of $X(z)$, i.e. $|X(z)| < \infty$, means

$$\begin{aligned} |X(z)| &= \left| \sum_n x[n] z^{-n} \right| \\ &\le \sum_n |x[n]| \left| z^{-n} \right| \\ &\le \sum_n |x[n]| |z|^{-n} < \infty \end{aligned}$$

  - This implies that convergence of the z-transform depends on $|z|$. That is, the ROC of $X(z)$ consists of all values of z such that the above inequality holds
- In general, if some value of z, say $z = z_1$, is in the ROC, then all values of z on the circle defined by $|z| = |z_1|$ are also in the ROC → ROC is a "ring".
- If ROC contains the unit circle, $|z| = 1$, then the FT of this sequence converges
- By its definition, $X(z)$ is a Laurent series (complex variable)
  - $X(z)$ is an *analytic function* in its ROC, i.e. $X(z)$ is complex differentiable in its ROC

DTFT vs. z-Transform: Examples 1 & 2

$$x_1[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Not absolutely summable but square summable (see Figure 2.11 in O&S), so DTFT converges in the mean-squared sense. However, *z*-Transform does not converge uniformly because $\sum_n x_1[n] r^{-n}$ is not absolutely summable for any value of *r*

$$x_2[n] = \cos \omega_0 n, \quad -\infty < n < \infty$$

- Not absolutely summable nor square summable. *z*-Transform does not converge uniformly because $\sum_n x_2[n] r^{-n}$ is not absolutely summable for any value of *r*. However, a "useful" DTFT (impulses) exists (using Dirac delta) since substituting $X_2(e^{j\omega})$ into the synthesis equation of the DTFT results in $x_2[n]$.

DTFT vs. z-Transform: Example 3

$$x_3[n] = a^n u[n], \quad |a| > 1, \quad -\infty < n < \infty$$

$$X_3(z) = \sum_n a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

For the z-Transform to exist, we require

$$\sum_{n=0}^{\infty} \left(az^{-1}\right)^n < \infty$$

For this to be true, we require $|az^{-1}| < 1$, or $|a| < |z|$. In this ROC, and using the relationship

$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$, $X_3(z)$ is

$$\sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad \text{for } |z| > |a|$$

Note: For sequences that are zero when $n < 0$, $X(z)$ involves only negative powers of z. Therefore, it is more convenient to express $X(z)$ as a function of z^{-1}, rather than z. However, note that factors such as $(1-az^{-1})^{-1}$ has both a pole and a zero.

DTFT vs. z-Transform: Example 4

$$x_4[n] = -a^n u[-n-1]$$

- The *z*-Transform is $$X_4(z) = -\sum_n a^n u[-n-1] z^{-n}$$ $$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$ $$= -\sum_{n=1}^{\infty} a^{-n} z^n$$ $$= 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n$$

The sequence will converge when $|a^{-1}z| < 1$ or $|z| < |a|$. So $X_4(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$

DTFT for $x_4[n]$ will not exist unless $|a| \geq 1$.

Some Common z-Transform Pairs

- $\delta[n] \leftrightarrow 1, \quad \delta\left[n-m\right] \leftrightarrow z^{-m}, \; m>0, \; |z|>0$

  $\delta\left[n+m\right] \leftrightarrow z^{m}, \; m>0, \; |z|<\infty$

- $u[n] \leftrightarrow \dfrac{1}{1-z^{-1}}, \quad |z|>1, \quad -u[-n-1] \leftrightarrow \dfrac{1}{1-z^{-1}}, \quad |z|<1$

- $a^n u[n] \leftrightarrow \dfrac{1}{1-az^{-1}}, \quad |z|>|a|$

  $-a^n u[-n-1] \leftrightarrow \dfrac{1}{1-az^{-1}}, \quad |z|<|a|$

- $r^n \cos\left[\omega_0 n\right] u[n] \leftrightarrow \dfrac{1-\left[r\cos\omega_0\right]z^{-1}}{1-\left[2r\cos\omega_0\right]z^{-1}+r^2 z^{-2}}, |z|>r$

- $r^n \sin\left[\omega_0 n\right] u[n] \leftrightarrow \dfrac{1-\left[r\sin\omega_0\right]z^{-1}}{1-\left[2r\sin\omega_0\right]z^{-1}+r^2 z^{-2}}, |z|>r$

Properties of ROC for *z*-Transform

- Most important and useful *z*-transforms are those in which $X(z)$ is a rational function inside the ROC.
- Rational functions $X(z) = \frac{P(z)}{Q(z)}$
  - **Poles** – roots of the denominator, the z such that $X(z) \rightarrow \infty$
  - **Zeroes** – roots of the numerator, the z such that $X(z) \rightarrow 0$

Properties of the ROC

(1) The ROC is a ring or disk in the z-plane centered at the origin

(2) The F.T. of $x[n]$ converges absolutely $\Leftrightarrow$ its ROC includes the unit circle

(3) The ROC cannot contain any poles

(4) If $x[n]$ is $finite-duration$, then the ROC is the entire z-plane except possibly at $z=0$ or $z=\infty$

Since the sequence is finite in length, it will always converge for any value of z. However, assuming the sequence starts from a negative value of n, then when $z=\infty$, z^{-n} will explode.

When the sequence is located in interval $n>0$, $x[n]z^{-n}$ will explode when $z=0$. Therefore, $z=0$ might not be included in the ROC

(5) If $x[n]$ is $right-sided$, the ROC, if exists, must be of the form $|z|>r_{\max}$ except possibly $z=\infty$, where $r_{\max}$ is the magnitude of the largest pole

In general, if $X(z)$ is rational, its inverse has the following form (assuming N poles: $\{d_k\}$) $x[n]=\sum_{k=1}^{N} A_k (d_k)^n$. (This can be seen later when we discussed using partial fraction expansion to compute the inverse z-transform of rational functions.) For a right-sided sequence, it means $n \geq N_1$, where N_1 is the first nonzero sample. The n^{th} term in the z-transform is $x[n]r^{-n}=\sum_{k=1}^{N} A_k \left(d_k r^{-1}\right)^n$.

This sequence converges if $\sum_{n=N_1}^{\infty} \left|d_k r^{-1}\right|^n < \infty$, $\left|d_k r^{-1}\right|<1$ or $|r|>|d_k|$, for every pole $k=1,\ldots,d_N$ is the outermost pole (which means that as n increases, the exponential above will grow the fastest), and d_N is the one with the largest absolute value, therefore, the ROC is outside d_N and extending to infinity

Properties of the ROC (contd) and System Stability

(6) If $x[n]$ is $left-sided$, the ROC, if exists, must be of the form $|z| < r_{\min}$ except possibly $z = 0$, where $r_{\min}$ is the magnitude of the smallest pole

Since the sequence is now extending to $-\infty$, that means $\left|d_k r^{-1}\right| > 1$, or $|r| < |d_k|$, in order for the sequence to converge. Therefore, the ROC is inside d_1 and extends inward toward zero

(7) If $x[n]$ is $two-sided$, the ROC must be of the form $r_1 < |z| < r_2$ if exists, where r_1 and r_2 are the magnitudes of the interior and exterior poles, respectively

(8) The ROC must be a connected region

- Stability and ROC
  - Systems that are stable are systems whose impulse responses are absolutely summable. This implies that the DTFT of the system will converge uniformly. In other words, the ROC has to include the unit circle because $\sum_n |h[n]| < \infty \iff \sum_n \left|h[n] z^{-n}\right| < \infty$, for $|z| = 1$. Therefore, for causal and stable systems, all the poles have to be inside the unit circle

Pole Location and Time-Domain Behavior for Causal Signals

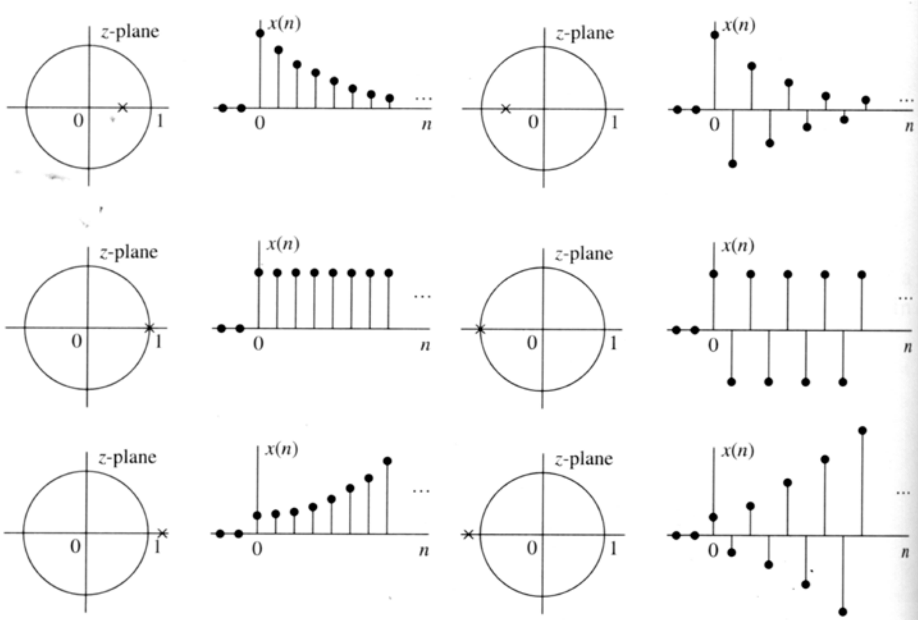

Figure 3.11 Time-domain behavior of a single-real pole causal signal as a function of the location of the pole with respect to the unit circle.

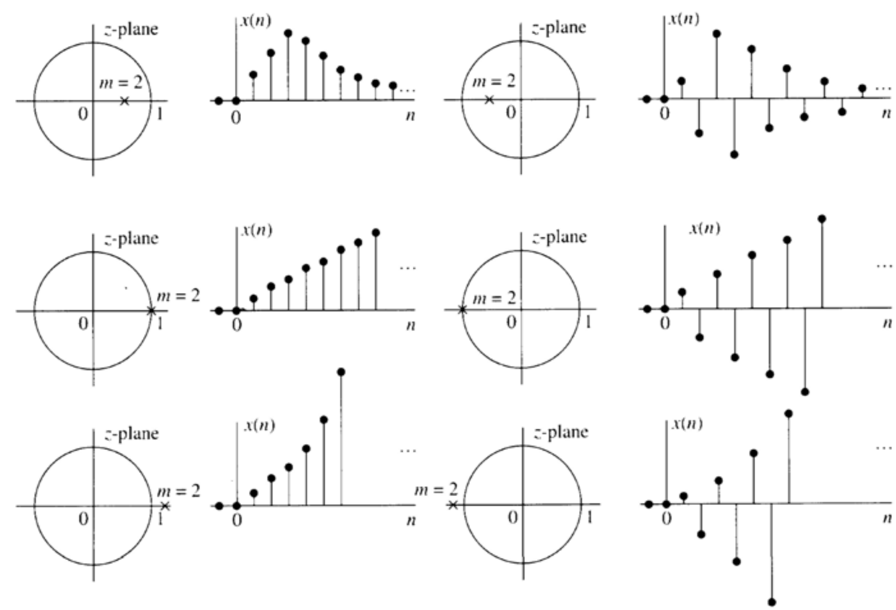

Figure 3.12 Time-domain behavior of causal signals corresponding to a double ($m = 2$) real pole, as a function of the pole location.

Pole Location and Time-Domain Behavior for Causal Signals

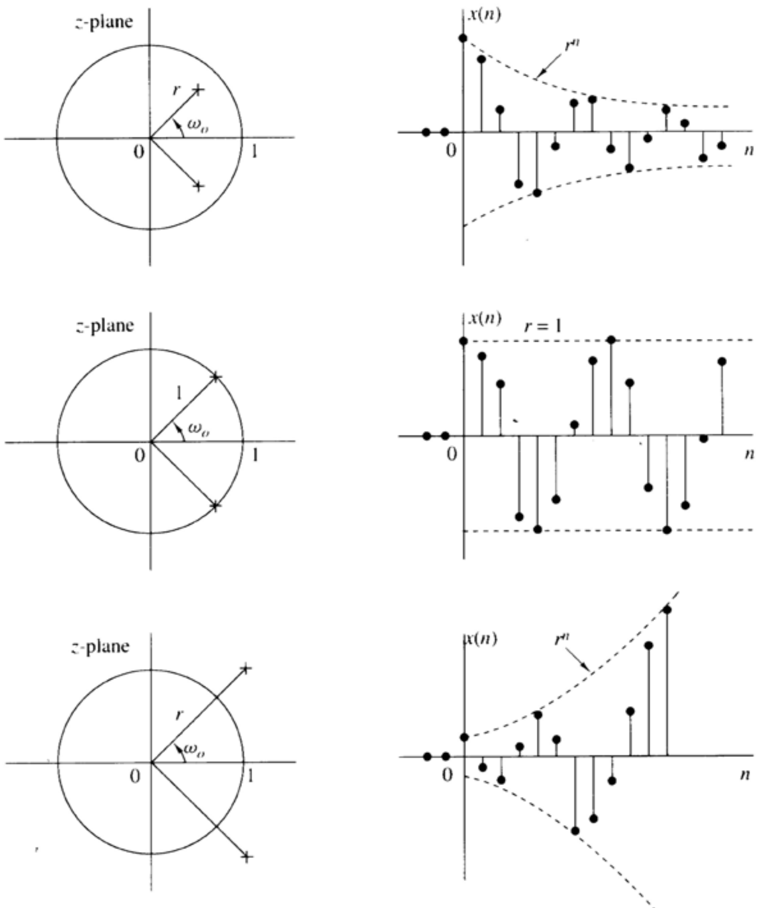

Figure 3.13 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

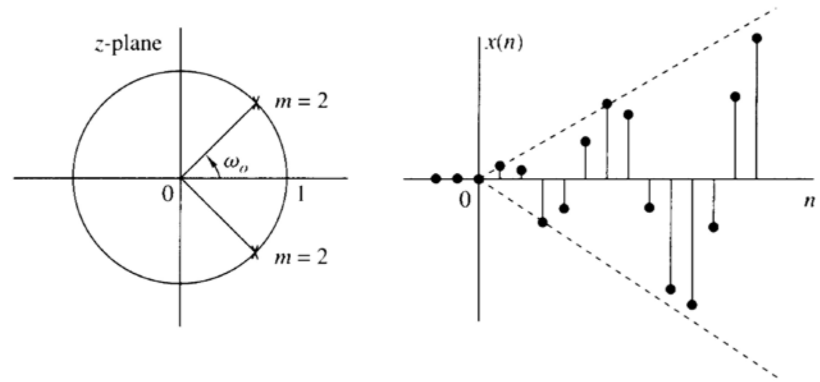

Figure 3.14 Causal signal corresponding to a double pair of complex-conjugate poles on the unit circle.

Inverse z-Transform

- Recall: Inverse: $x[n] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz \equiv Z^{-1}[X(z)]$
- This formula can be proved using Cauchy integral theorem (complex variable theory)
- Methods of evaluating the inverse z-transform
  1. Table lookup or inspection (from entries in table)
  2. Partial fraction expansion
  3. Power series expansion

Partial Fraction Expansion

$$X(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots+a_Nz^{-N}}\rightarrow X(z)=\frac{z^N(b_0z^M+\cdots+b_M)}{z^M(a_0z^N+\cdots+a_N)}$$

Hence, it has M zeros (roots of $\sum_{k=0}^{M}b_kz^{M-k}$), N poles (roots of $\sum_{k=0}^{N}a_kz^{N-k}$),

and $\left(M-N\right)$ poles at zero if $M>N$ (or $\left(N-M\right)$ zeros at zero if $N>M$)

$$\rightarrow X(z)=\frac{b_0(1-c_1z^{-1})\cdots(1-c_Mz^{-1})}{a_0(1-d_1z^{-1})\cdots(1-d_Nz^{-1})};$$

c_k nonzero zeros; d_k nonzero poles

Partial Fraction Expansion

- **Case 1**: $M < N$, strictly proper

  Simple (single) poles:

$$X(z) = \frac{A_1}{(1-d_1 z^{-1})} + \frac{A_2}{(1-d_2 z^{-1})} + \cdots + \frac{A_N}{(1-d_N z^{-1})}, \quad \text{where } A_k = \left(1-d_k z^{-1}\right) X\left(z\right)\Big|_{z=d_k}$$

  Multiple poles: Assume d_i is the s^{th} order pole (repeated s times)

$$X(z) = \underbrace{\sum_{k=1,k\neq i}^{N} \frac{A_k}{(1-d_k z^{-1})}}_{\text{single-pole term}} + \underbrace{\frac{C_1}{(1-d_i z^{-1})} + \frac{C_2}{(1-d_i z^{-1})^2} + \cdots + \frac{C_s}{(1-d_i z^{-1})^s}}_{\text{multiple-pole terms}},$$

$$\text{where } C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1-d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}$$

- **Case 2**: $M \geq N$

$$X(z) = \underbrace{\sum_{r=0}^{M\text{-}N} B_r z^{\text{-}r}}_{\text{impulse}} + \underbrace{\sum_{k=1,k\neq i}^{N} \frac{A_k}{(1-d_k z^{-1})}}_{\text{single-pole terms}} + \underbrace{\sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}}_{\text{multiple-poles terms}}$$

Non-Repeated Poles Example

Example: $X(z) = \dfrac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \quad |z| > \frac{1}{2}$

$$X(z) = \frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$A_1 = \left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}} = -1$$

$$A_2 = \left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = 2$$

$$\Rightarrow X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$\Leftrightarrow x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

Repeated Poles Example

$$\text{Example:} \quad H(z) = \frac{1+3z^{-1}}{\left(1+2z^{-1}\right)^2\left(1+5z^{-1}\right)}, \quad |z| > |a|$$

$$H(z) = \frac{A_1}{\left(1+2z^{-1}\right)} + \frac{A_2}{\left(1+2z^{-1}\right)^2} + \frac{A_3}{\left(1+5z^{-1}\right)}, \quad |z| > |a|$$

$$A_2 = \left(1+2z^{-1}\right)^2 H(z)\Big|_{z=-2} = \left.\frac{1+3z^{-1}}{\left(1+5z^{-1}\right)}\right|_{z=-2} = \frac{1}{3}$$

$$A_3 = \left(1+5z^{-1}\right) H(z)\Big|_{z=-5} = \left.\frac{1+3z^{-1}}{\left(1+2z^{-1}\right)^2}\right|_{z=-5} = \frac{10}{9}$$

$$A_1 = \left[\frac{d}{dz^{-1}}\left(1+2z^{-1}\right)^2 H(z)\right]\Bigg|_{z=-2}$$

$$= \left[\frac{d}{dz^{-1}}\frac{1+3z^{-1}}{\left(1+5z^{-1}\right)}\right]\Bigg|_{z=-2} = \frac{\left(1+5z^{-1}\right)3-\left(1+3z^{-1}\right)5}{\left(1+5z^{-1}\right)^2} = -\frac{8}{9}$$

Power Series Expansion

Case 1: Right-sided sequence, ROC: $|z| > r_{\max}$

E.g. $X(z) = \dfrac{1}{1-az^{-1}}, \quad |z| > |a|$

$$
\begin{array}{r l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
1-az^{-1} \,\big) & 1 \\
 & 1 - az^{-1} \\
 & \text{-------------} \\
 & \quad az^{-1} \\
 & \quad az^{-1} - a^2z^{-2} \\
 & \quad \text{--------} \\
 & \qquad\quad a^2z^{-2} \quad \cdots
\end{array}
$$

Therefore, $x[n] = a^n u[n]$

Power Series Expansion

Case 2: Left-sided sequence, ROC: $|z| < r_{\min}$

It is expanded in powers of z

E.g. $X(z) = \dfrac{1}{1 - az^{-1}}, \quad |z| < |a|$

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^2 - \cdots \\
\hline
-a + z & z \\
 & z - a^{-1}z^2 \\
 & \text{-----------} \\
 & \quad a^{-1}z^2
\end{array}
$$

Therefore, $x[n] = -a^n u[-n-1]$.

Power Series Expansion

Case 3: Two-sided sequence, ROC: $r_1 < |z| < r_2$

$$X(z) = \underbrace{X_+(z)}_{\text{converges for } |z|>r_1} + \underbrace{X_-(z)}_{\text{converges for } |z|<r_2}$$

$$\rightarrow x[n] = \underbrace{x_+[n]}_{\text{casual sequence}} + \underbrace{x_-[n]}_{\text{anti-causal sequence}}$$

z-Transform Properties

If $x[n] \Leftrightarrow X(z)$ and $y[n] \Leftrightarrow Y(z)$, ROC: R_x, R_Y

Linearity:

$$ax[n]+by[n] \leftrightarrow aX(z)+bY(z)$$

ROC: $R' \supset R_X \cap R_Y$ -- At least as large as their intersection; larger if pole/zero cancellation occurs

Time Shifting:

$$x[n-n_0] \leftrightarrow z^{-n_0}X(z) \qquad \text{ROC: } R' = R_X \pm \{0 \text{ or } \infty\}$$

Multiplication by an exponential sequence:

$$a^n x[n] \leftrightarrow X(z/a) \qquad \text{ROC: } R' = |a| R_X$$ -- expands or contracts

Differentiation of $X(z)$:

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}, \qquad \text{ROC: } R' = R_X$$

Conjugation of a complex sequence:

$$x^*[n] \leftrightarrow X^*(z^*), \qquad \text{ROC: } R' = R_X$$

z-Transform Properties

Time Reversal:

$$x^*[-n] \leftrightarrow X^*\left(\frac{1}{z^*}\right)$$

ROC: $R' = 1/R_X$ (Meaning: If $R_X: 1/r_R < |z| < 1/r_L$, then $R': 1/r_L < |z| < 1/r_R$

Corollary: $x[-n] \leftrightarrow X\left(\frac{1}{z}\right)$

Convolution:

$x[n] * y[n] \leftrightarrow X(z)Y(z)$ ROC: $R' \supset R_X \cap R_Y$ (=, if no pole/zero cancellation)

Initial Value Theorem:

If $x[n] = 0,\ n < 0$

then $x[0] = \lim_{z\to\infty} X(z)$

Final Value Theorem:

If (1) $x[n] = 0,\ n < 0$, and

(2) all singularities of $\left(1 - z^{-1}\right)X(z)$ are inside the unit circle,

then $x[\infty] = \lim_{z\to 1}\left(1 - z^{-1}\right)X(z)$

Remarks: (1) If all poles of $X(z)$ are inside the unit circle, $x[n] \to 0$ as $n \to \infty$

(2) If there are multiple poles at "1", $x[n] \to \infty$ as $n \to \infty$

(3) If poles are on the unit circle but not at "1", $x[n] \cong \cos\omega_0 n$

z-Transform Solutions of Linear Difference Equations

Using ***single - sided*** z-transform:

$$Z\{y[n-1]\} = z^{-1}Y(z) + y[-1]$$
$$Z\{y[n-2]\} = z^{-2}Y(z) + z^{-1}y[-1] + y[-2]$$
$$Z\{y[n-3]\} = z^{-3}Y(z) + z^{-2}y[-1] + z^{-1}y[-2] + y[-3]$$

For causal signals, their single-sided z-transforms are identical to their two-sided z-transforms

E.g. Find $y[n]$ of the difference equation

$$y[n] - 0.5y[n-1] = x[n] \text{ with } x[n] = 1,\ n \geq 0, \text{ and } y[-1] = 1$$

Take the single-sided z-transform of the above equation, we get

$$\Rightarrow Y(z) - 0.5\left\{z^{-1}Y(z) + y[-1]\right\} = X(z) = \frac{1}{1-z^{-1}}$$

$$\Rightarrow Y(z) = \left\{\frac{1}{1-0.5z^{-1}}\right\}\left\{0.5 + \frac{1}{1-z^{-1}}\right\} = \frac{0.5}{1-0.5z^{-1}} + \frac{1}{\left(1-0.5z^{-1}\right)\left(1-z^{-1}\right)}$$

$$\Rightarrow Y(z) = \frac{2}{1-z^{-1}} - \frac{0.5}{1-0.5z^{-1}}$$

Take the inverse z-transform

$$\Rightarrow y[n] = 2 - 0.5(0.5)^n, \quad n \geq 0$$