Transform Analysis of LTI Systems

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Outline

- Frequency response of LTI systems
- Linear constant-coefficient difference equation
- Magnitude and phase
- Minimum phase systems
- Linear phase systems

Frequency Response of LTI Systems

- An LTI can completely be characterized in the time domain by its *impulse response* (assuming no initial conditions)
- Time: $y[n] = x[n]*h[n]$
- z-transform: $Y(z) = X(z)H(z)$
  - $H(z)$ is the system function or transfer function
- Magnitude or gain

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = H(z)H^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j\omega}} = H_R^2\left(e^{j\omega}\right) + H_I^2\left(e^{j\omega}\right)$$

- Phase response or phase shift

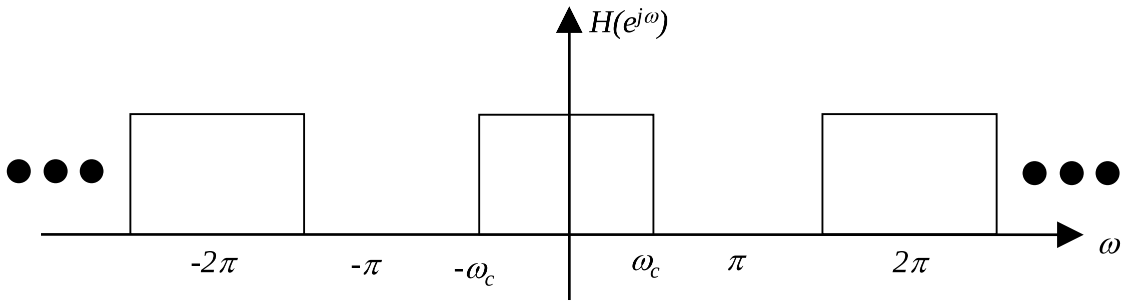
$$\measuredangle H\left(e^{j\omega}\right) = \tan^{-1}\left(\frac{H_I\left(e^{j\omega}\right)}{H_R\left(e^{j\omega}\right)}\right)$$

  - Be careful of which quadrant you are referring to (more later)

Frequency Response of Ideal Lowpass Filter

Ideal lowpass: $h_{lp}[n] = \dfrac{\sin(\omega_c n)}{\pi n}$

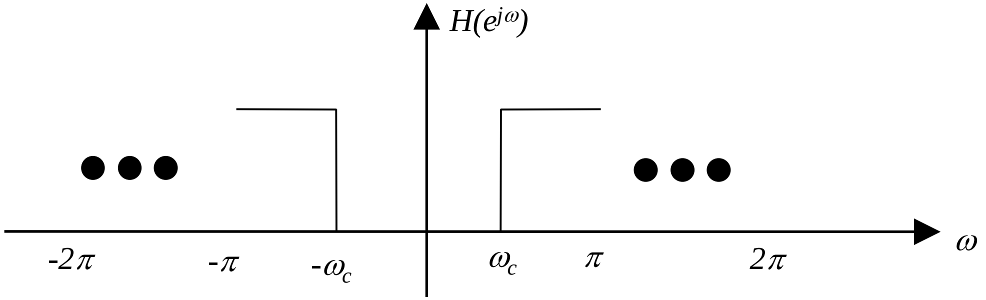
$$H_{LP}\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$



Frequency Response of Ideal Highpass Filter

Ideal highpass: $h_{hp}[n] = \delta[n] - \dfrac{\sin(\omega_c n)}{\pi n}$

$$H_{HP}\left(e^{j\omega}\right) = \begin{cases} 1, & \omega_c \le |\omega| \le \pi \\ 0, & \text{otherwise} \end{cases}$$



Remark: This definition includes the phase specification. That is, zero phase for all frequencies. Practical physical systems cannot achieve this specification. In addition, it is noncausal and needs infinite input samples to compute the current output ➔ they are not *computationally realizable*. Causal approximations to ideal frequency-selective filters must have a nonzero phase response (more on this later).

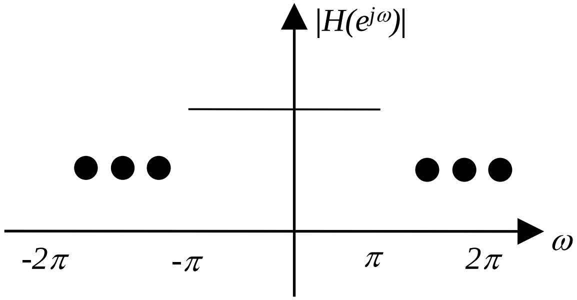
Typical filters include: lowpass, highpass, bandpass,bandstop, all-pass filters

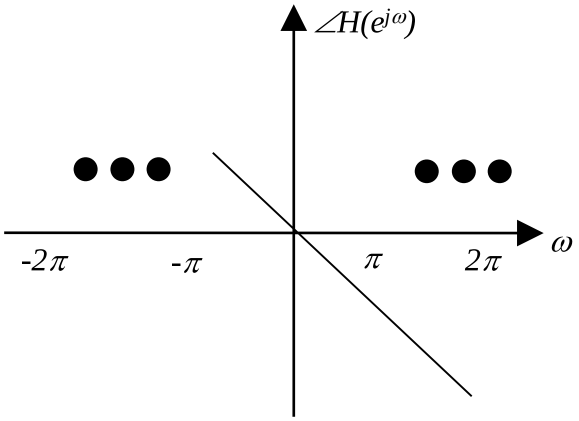
Frequency Response of Phase Delay

$$h_{id}[n]=\delta[n-n_d]\Leftrightarrow H_{id}\left(e^{j\omega}\right)=e^{-j\omega n_d}$$

The output is a delayed version of the input. Shape of the input waveform is not changed

$$\begin{cases} \left|H_{id}\left(e^{j\omega}\right)\right|=1 \\ \angle H_{id}\left(e^{j\omega}\right)=-\omega n_d, \quad |\omega|<\pi \end{cases}$$





Linear Phase and Group Delay

Linear phase: The phase response is a linear function of ω (passing through the origin). A frequency-selective filter with a linear phase is often acceptable and can be approximated by a practical system. That is

$$H\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega n_d}, & \omega_1 \le |\omega| \le \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

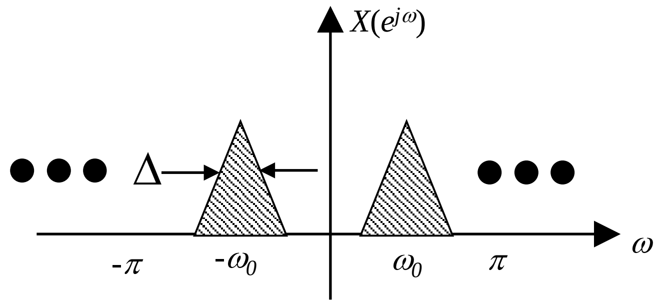
Group delay: $$\tau(\omega) \triangleq grd\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\} = n_d$$

Group delay is a convenient measure of the linearity of the phase. The basic property of group delay relates to the effect of the phase on a narrowband signal. It is clear for the ideal delay, $\tau(\omega) = n_d$, is a constant (independent of ω).

Narrowband Signals in Communication Systems

Consider the output of a system $H\left(e^{j\omega}\right)$ for a narrowband signal

$$x[n] = s[n]\cos(\omega_0 n)$$
$$= \frac{1}{2}s[n]e^{j\omega_0 n} + \frac{1}{2}s[n]e^{-j\omega_0 n}$$
$$\Leftrightarrow X\left(e^{j\omega}\right) = \frac{1}{2}S\left(e^{j(\omega+\omega_0)}\right) + \frac{1}{2}S\left(e^{j(\omega-\omega_0)}\right).$$



Narrowband signal: $\Delta << \omega_0$ and $S(e^{j\omega}) = 0$ for $|\omega| > \Delta$ and $X(e^{j\omega})$ is narrowband around $\pm\omega_0$

It is assumed that $x[n]$ is nonzero only around $\omega = \omega_0 \Rightarrow$ the effect of the phase of the system can be approximated around $\omega = \omega_0$ as the linear approximation. Let

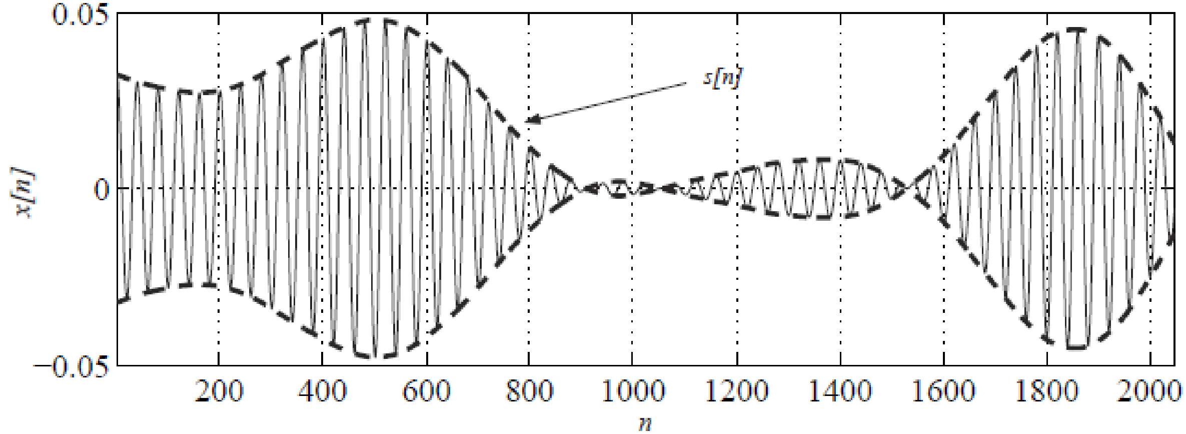
$$\measuredangle H\left(e^{j\omega}\right) = -\phi_0 - \omega n_d \;\Rightarrow\; H\left(e^{j\omega}\right) = \begin{cases} e^{j(-\omega n_d + \phi_0)}, & \omega < 0, \\ e^{j(-\omega n_d - \phi_0)}, & \omega \geq 0. \end{cases} \text{ (assuming } \left|H\left(e^{j\omega}\right)\right| = 1\text{)}$$

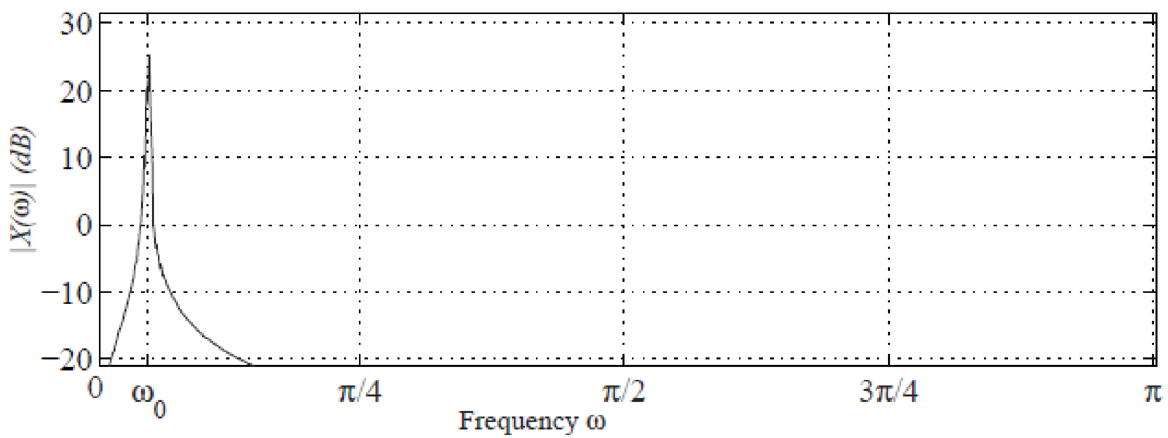
$$\Rightarrow Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right) = \frac{1}{2}e^{-j\phi_0}e^{-j\omega n_d}S\left(e^{j(\omega-\omega_0)}\right) + \frac{1}{2}e^{j\phi_0}e^{-j\omega n_d}S\left(e^{j(\omega+\omega_0)}\right)$$

Note that: $\delta[n-n_d] \;\Leftrightarrow\; e^{-j\omega n_d}$, $\cos(\omega_0 n + \phi) \Leftrightarrow \pi e^{j\phi}\delta(\omega-\omega_0) + \pi e^{-j\phi}\delta(\omega+\omega_0)$ (one cycle)

$y[n] = \mathcal{F}\{h[n]\} * \mathcal{F}\{x[n]\}$

Narrowband Signals





$s[n]$ is a slowly-varying envelope in $x[n]$.

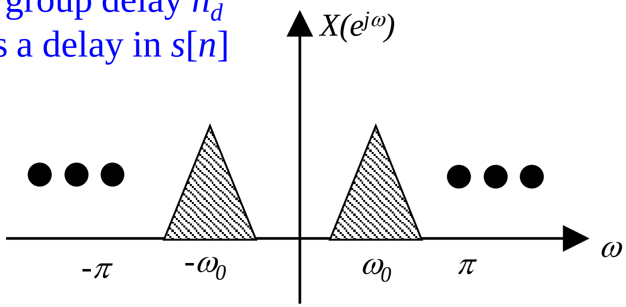
Narrowband Signals in Communication Systems

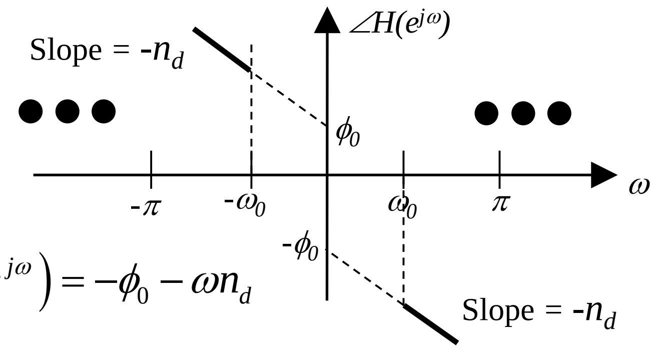
so that the output can be approximated as

$$y[n] = \delta[n-n_d] * \frac{1}{2}\left(s[n]e^{j(\omega_0 n-\phi_0)} + s[n]e^{-j(\omega_0 n-\phi_0)}\right)$$
$$= \delta[n-n_d] * s[n]\cos(\omega_0 n-\phi_0)$$
$$= s[n-n_d]\cos(\omega_0 n-\omega_0 n_d-\phi_0)$$

$$Y(e^{j\omega}) = \frac{1}{2}e^{-j\phi_0}e^{-j\omega n_d}S\left(e^{j(\omega-\omega_0)}\right) + \frac{1}{2}e^{j\phi_0}e^{-j\omega n_d}S\left(e^{j(\omega+\omega_0)}\right)$$

Notice the group delay n_d contributes a delay in $s[n]$





$$\measuredangle H(e^{j\omega}) = -\phi_0 - \omega n_d$$

We see that the time delay of the envelope $s[n]$ of the narrowband signal $x[n]$ with Fourier transform centered at ω_0 is given by the negative of the slope of the phase at ω_0. The deviation of the group delay from a constant indicates the degree of nonlinearity of the phase.

Remark: arg[H(.)]: *continuous phase* (|value| < π or > π)

ARG[H(.)]: *principal value* (|value| < π)

Example

Consider a filter with group delay and frequency response magnitude shown in Figure 5.1a and b, respectively

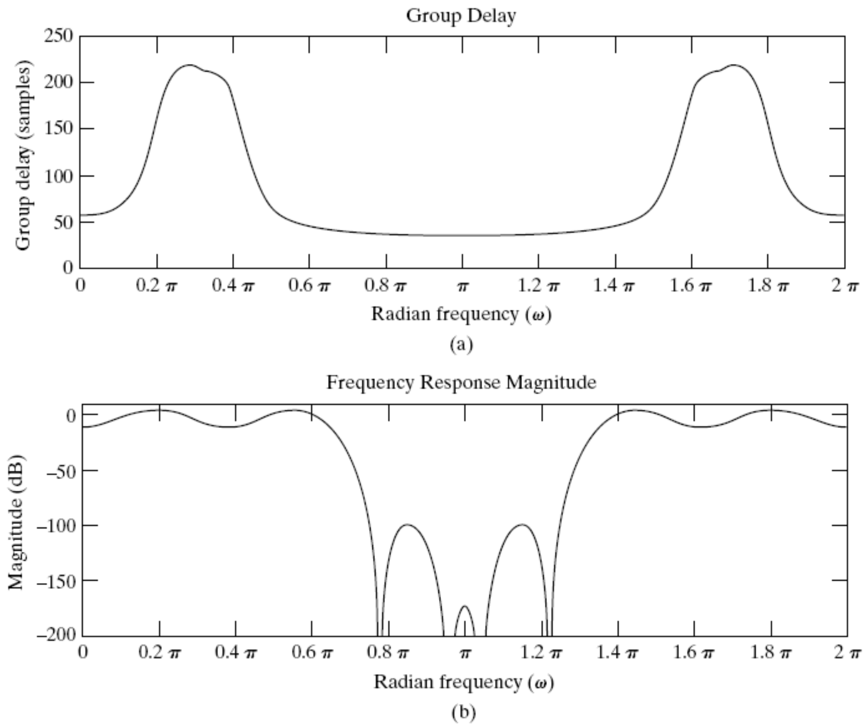


Figure 5.1 Frequency response magnitude and group delay for the filter in Example 5.1.

Example (contd)

Figure 5.2, shows an input signal and its spectrum. Note that the input consists of three consecutive narrowband pulses, at frequencies $\omega = 0.85\pi$, $\omega = 0.25\pi$, $\omega = 0.5\pi$.

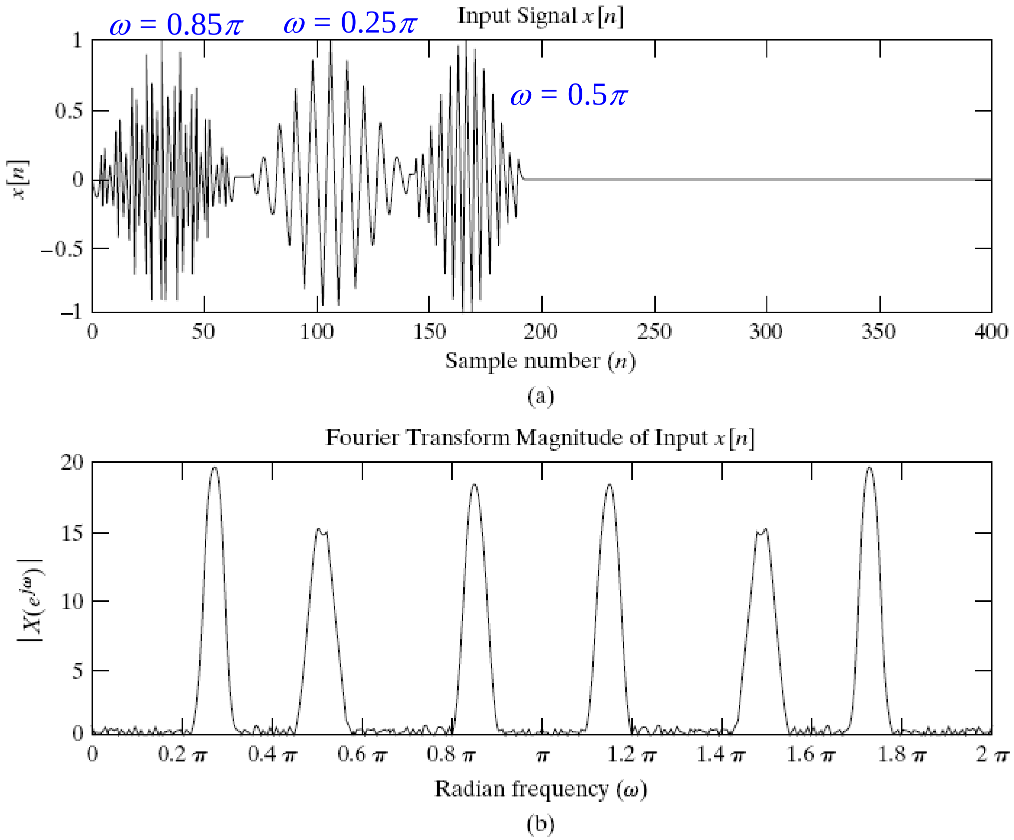


Figure 5.2 Input signal and associated Fourier transform magnitude for Example 5.1.

Example (contd)

In Figure 5.3, it shows the resulting output signal.

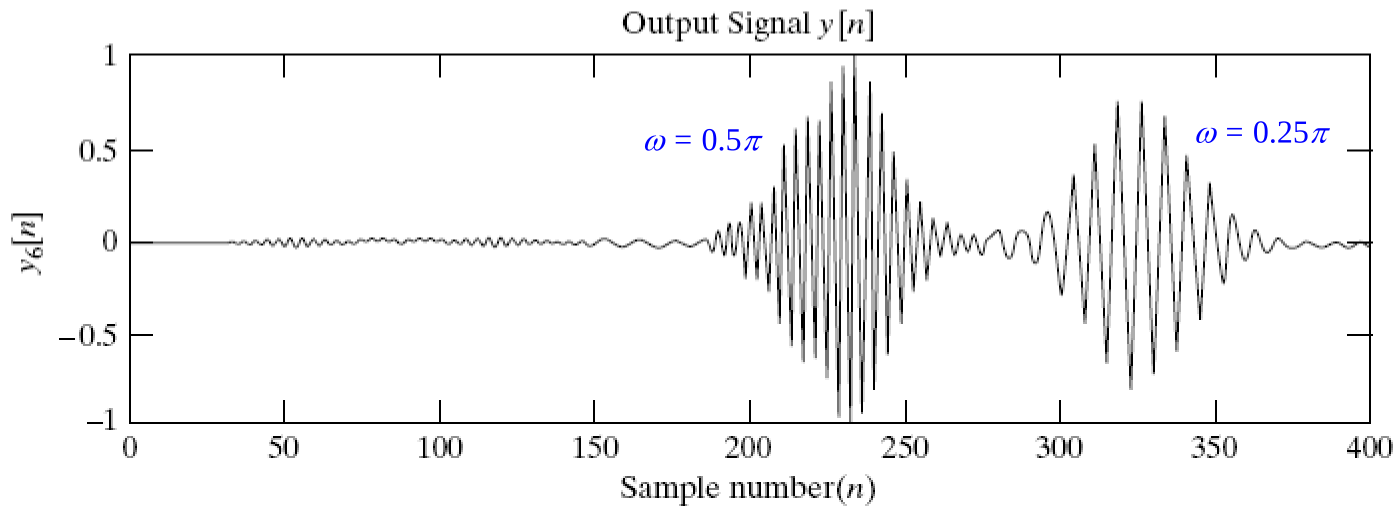


Figure 5.3 Output signal for Example 5.1.

Since the filter has considerable attenuation at $\omega = 0.85\pi$, the pulse at that frequency is not clearly present in the output. Also, since the group delay at $\omega = 0.25\pi$ is approximately 200 samples and at $\omega = 0.5\pi$ is approximately 50 samples, the second pulse in $x[n]$ is delayed by about 200 samples and the third pulse by 50 samples.

Systems Described by Linear Constant Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$\Rightarrow$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}$$

Note:

- $X(z)$ and $Y(z)$ have overlapping regions of convergence.
- $(1-c_k z^{-1})$ contributes a zero at $z=c_k$ and a pole at $z = 0$.
- $(1-d_k z^{-1})$ contributes a pole at $z=d_k$ and a zero at $z = 0$.

BIBO Stability and Causality (Review)

BIBO Stability

Recall that if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (absolutely summable), then $h[n]$ is BIBO stability. This is equivalent to the condition that $\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$ for $|z| = 1$. This implies that the ROC of $H(z)$ include the unit circle.

Causality

Causal (right-sided sequence) iff ROC: $|z| > r_{max}$

Causal and Stable

All poles are inside the unit circle

Example:

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

$$\Rightarrow H(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$

Inverse Systems

$$X(z) \rightarrow \boxed{H(z)} \rightarrow \boxed{H_i(z)} \rightarrow X(z)$$

$$H(z)H_i(z) = 1 \quad \text{or} \quad H(z) = \frac{1}{H_i(z)}$$

So poles of $H(z)$ → zeros of $H_i(z)$
So zeros of $H(z)$ → poles of $H_i(z)$
$H_i(z)$ is *causal and stable* if all zeros of $H(z)$ are inside the unit circle

Minimum phase
Both poles and zeros are inside the unit circle (more on this later)

Impulse Response of Rational System Functions (IIR)

IIR (infinite impulse response) :

Assuming that $M > N$, then $H(z)$ is

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

First term is obtained by long division and 2^{nd} term can be obtained using partial fraction expansion. The impulse response can thus be written as

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n],$$

where the first term is included only if $M \geq N$. When there is at least one non-zero pole $\{d_k\}$ that is not canceled by a zero, there will be at least one term of the form $A_k (d_k)^n u[n]$, and thus the impulse response will not be of finite length. This is known infinite impulse response system or IIR.

Impulse Response of Rational System Functions (FIR)

FIR (finite impulse response) :

When there is no poles except at $z=0$, i.e. $N=0$, then $H(z)$ is of the form

$$H(z)=H(z)=\sum_{k=1}^{M} b_k z^{-k}.$$

Then

$$h[n]=\sum_{k=0}^{M} b_k \delta[n-k]=\begin{cases} b_n, & 0\le n\le M, \\ 0, & \text{otherwise.} \end{cases}$$

Since the impulse is of finite length, this system is known as finite impulse response or FIR.

FIR Example

Example: $$h[n] = \begin{cases} a^n, & 0 \le n \le M, \\ 0, & otherwise. \end{cases}$$

Then the system function is

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}.$$

We can see that the zeros are located at

$$z_k = ae^{j\frac{2\pi k}{M+1}}, \qquad k = 0, 1, \ldots, M,$$

which is the $(M+1)^{th}$ root of unity. a is assumed to be real and positive, and the poles at $z = a$ is canceled by a zero.

$M = 7$

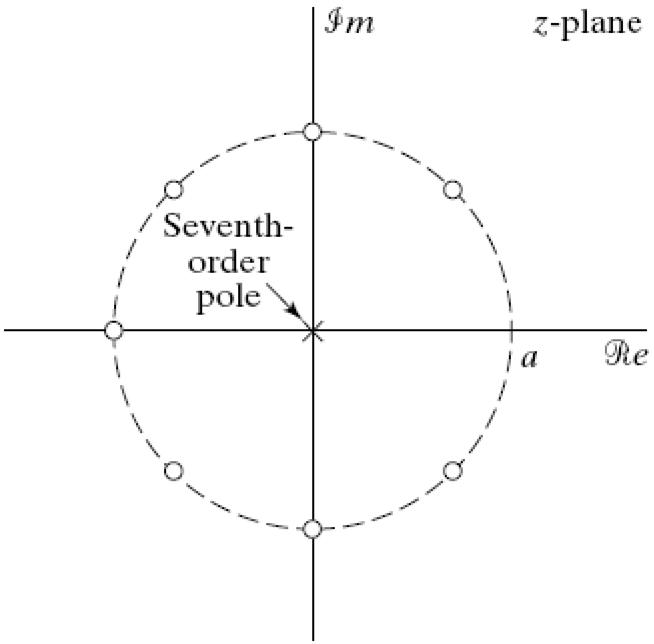


Figure 5.6 Pole–zero plot for Example 5.7.

Frequency Response for Rational System Functions: Magnitude

Assuming that $H\left(e^{j\omega}\right)$ is a rational function, i.e.

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})}.$$

Then

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|1-c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1-d_k e^{-j\omega}\right|},$$

where $|\bullet|$ denotes **magnitude** NOT absolute value. Hence, the squared magnitude of $H\left(e^{j\omega}\right)$ is

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right) H^*\left(e^{j\omega}\right) = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M}\left(1-c_k e^{-j\omega}\right)\left(1-c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1-d_k e^{-j\omega}\right)\left(1-d_k^* e^{j\omega}\right)}.$$

Frequency Response for Rational System Functions: Magnitude

This is equivalent to

$$\left|H\left(e^{j\omega}\right)\right|^2=\left|H\left(z\right)\right|^2\Big|_{z=e^{j\omega}}=H\left(z\right)H^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j\omega}}=\left|\frac{b_0}{a_0}\right|^2\left|\frac{\prod_{k=1}^{M}\left(1-c_kz^{-1}\right)\left(1-c_k^*z\right)}{\prod_{k=1}^{N}\left(1-d_kz^{-1}\right)\left(1-d_k^*z\right)}\right|_{z=e^{j\omega}}.$$

For convenience, we sometimes express the magnitude in terms of the $\log$ magnitude which has the unit of *decibels* or dB :

$$20\log_{10}|H(e^{j\omega})|=20\log_{10}\left|\left(\frac{b_0}{a_0}\right)\right|+\sum_{k=1}^{M}20\log_{10}\left|1-c_ke^{-j\omega}\right|-\sum_{k=1}^{N}20\log_{10}\left|1-d_ke^{-j\omega}\right|.$$

Clearly, $20\log_{10}|Y(e^{j\omega})|=20\log_{10}|H(e^{j\omega})|+20\log_{10}|X(e^{j\omega})|$

Frequency Response for Rational System Functions: Phase and Group Delay

Phase:

$$\angle H(e^{j\omega}) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} \angle\left(1 - c_k e^{-j\omega}\right) - \sum_{k=1}^{N} \angle\left(1 - d_k e^{-j\omega}\right)$$

Group delay :

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\angle H\left(e^{j\omega}\right) = \sum_{k=1}^{N} \frac{d}{d\omega}\left(\arg\left[1 - d_k e^{-j\omega}\right]\right) - \sum_{k=1}^{M} \frac{d}{d\omega}\left(\arg\left[1 - c_k e^{-j\omega}\right]\right)$$

Phase Ambiguity

Problem? Phase ambiguity. The real filter phase is continuous and can be greater than π or smaller than $-\pi$. However, the calculated phase is the principal value,

$$-\pi < \mathrm{ARG}[H(e^{j\omega})] < \pi$$

which is what we get if we use the arctangent (atan or atan2 in Matlab) function, this is

$$ARG\left[H\left(e^{j\omega}\right)\right] = \arctan\left[\frac{H_I\left(e^{j\omega}\right)}{H_R\left(e^{j\omega}\right)}\right],$$

where $H_R\left(e^{j\omega}\right)$ and $H_I\left(e^{j\omega}\right)$ are real and imaginary parts of $H\left(e^{j\omega}\right)$, respectively.

Therefore, $\angle H\left(e^{j\omega}\right) = ARG\left[H\left(e^{j\omega}\right)\right] + 2\pi r(\omega)$, where $r(\omega)$ is a positive or negative is a function of ω. See Figure 5.7 (right). Using $ARG[\bullet]$, then

$$ARG\left[H\left(e^{j\omega}\right)\right] = ARG\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} ARG\left[1-c_k e^{-j\omega}\right] - \sum_{k=1}^{N} ARG\left[1-d_k e^{-j\omega}\right] + 2\pi r(\omega).$$

The last term is needed because the principal value of a sum of angles is not equal to the sum of the principal values of the individual angles.

- The principal angle of $H(e^{j\omega})$ can still be computed by computing the principal angle of each pole and zero and an appropriate multiple of 2π can be added or subtracted to obtain the principal value of the total phase function

➔ ARG has jumps of 2π due to the integer multiples of 2π that must be subtracted in certain regions to bring the phase curve within the range of the principal value

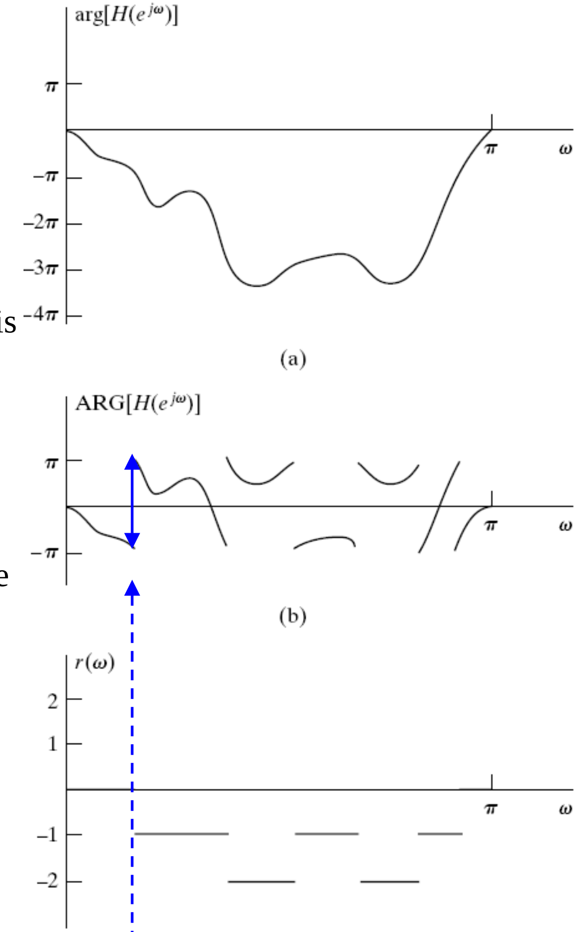


Figure 5.7 (a) Continuous-phase curve for a system function evaluated on the unit circle. (b) Principal value of the phase curve in part (a). (c) Integer multiples of 2π to be added to $\mathrm{ARG}[H(e^{j\omega})]$ to obtain $\arg[H(e^{j\omega})]$.

Group Delay

- The group delay function $\tau(\omega)$ is still the derivative of the continuous phase function (not principal value)

$$\tau(\omega) = \text{grd}\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\}.$$

- Except at the discontinuities of $\text{ARG}\left[H\left(e^{j\omega}\right)\right]$,

$$\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\} = \frac{d}{d\omega}\left\{\text{ARG}\left[H\left(e^{j\omega}\right)\right]\right\}$$

- Can also represent $\tau(\omega)$ using ambiguous phase $\measuredangle H\left(e^{j\omega}\right)$ as

$$\tau(\omega) = -\frac{d}{d\omega}\left[\measuredangle H\left(e^{j\omega}\right)\right]$$

with the interpretation that impulses caused by discontinuities of size 2π in $\angle H\left(e^{j\omega}\right)$ are ignored.

Example: Single Zero or Pole

Given a single zero system: $H(z) = \left(1 - c_0 z^{-1}\right)$, zero: $c_0 = re^{j\theta}$

Geometric evaluation: $H\left(e^{j\omega}\right) = \left(1 - c_0 e^{-j\omega}\right) = \dfrac{e^{j\omega} - c_0}{e^{j\omega}}$

$= \dfrac{\mathbf{v}_3}{\mathbf{v}_1}$ (see Fig. 5.9)

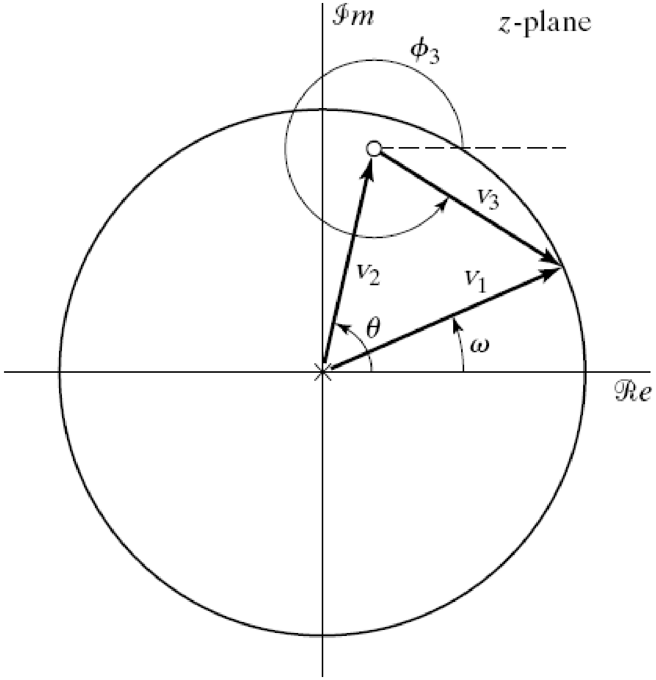


Figure 5.9 z-plane vectors for a first-order system function evaluated on the unit circle, with $r < 1$.

$$\left|H(z)\right|^2 = H(z) H^*\left(\frac{1}{z^*}\right) = \left(1 - c_0 z^{-1}\right)\left(1 - c_0^* z\right)$$

$$= 1 - c_0 z - c_0^* z^{-1} + \left|c_0\right|^2$$

Let $z = e^{j\omega}$ and using Euler's identity

$$\left|H\left(e^{j\omega}\right)\right|^2 = 1 - c_0\left(\cos\omega + j\sin\omega\right) - c_0^*\left(\cos\omega - j\sin\omega\right) + \left|c_0\right|^2$$

$$= 1 - 2\,\mathrm{Re}\left[c_0\right]\cos\omega - 2\,\mathrm{Im}\left[c_0\right]\sin\omega + \left|c_0\right|^2$$

$$= 1 - 2\left(\mathrm{Re}\left[c_0\right]\cos\omega + \mathrm{Im}\left[c_0\right]\sin\omega\right) + \left|c_0\right|^2.$$

Assume the location of the zero is $c_0 = re^{j\theta}$, we have

$$\left|H\left(e^{j\omega}\right)\right|^2 = 1 - 2r\left(\cos\omega\cos\theta + \sin\omega\sin\theta\right) + r^2$$

$$= 1 - 2r\cos\left(\omega - \theta\right) + r^2.$$

Note: The log magnitude can be written as

$$20\log_{10}\left|H\left(e^{j\omega}\right)\right| = 10\log_{10}\left|H\left(e^{j\omega}\right)\right|^2$$

$$= 10\log_{10}\left(1 - 2r\cos\left(\omega - \theta\right) + r^2\right)$$

$$= 20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right|.$$

Example: Single Zero or Pole (contd)

Phase: $\angle H(e^{j\omega}) = \angle\left(1 - c_0 e^{-j\omega}\right) = \angle\left(e^{j\omega} - c_0\right) - \angle e^{j\omega}.$

Substitute $c_0 = re^{j\theta}$, we have

$$
\begin{aligned}
\angle H(e^{j\omega}) &= \angle(1 - re^{j\theta}e^{-j\omega}) = \angle\left(1 - r\cos(\omega-\theta) + jr\sin(\omega-\theta)\right) \\
&= \tan^{-1}\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)} \\
&= \angle\left(e^{j\omega} - re^{j\theta}\right) - \angle e^{j\omega} \\
&= \angle \mathbf{v}_3 - \angle \mathbf{v}_1 \quad \text{(see Fig. 5.9)} \\
&= \phi_3 - \phi_1 \\
&= \phi_3 - \omega. \qquad \text{(see Fig. 5.9)}
\end{aligned}
$$

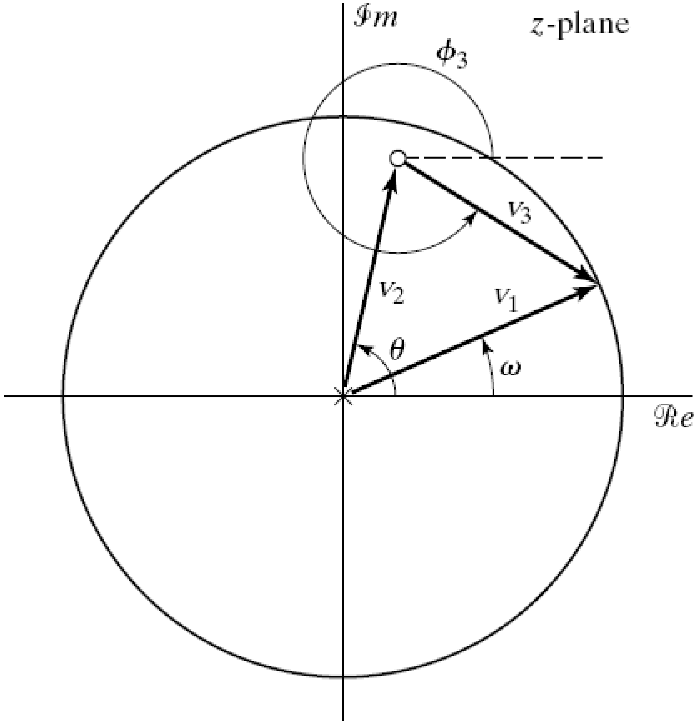


Figure 5.9 *z*-plane vectors for a first-order system function evaluated on the unit circle, with $r < 1$.

Frequency response of the single zero system by varying the parameter r.

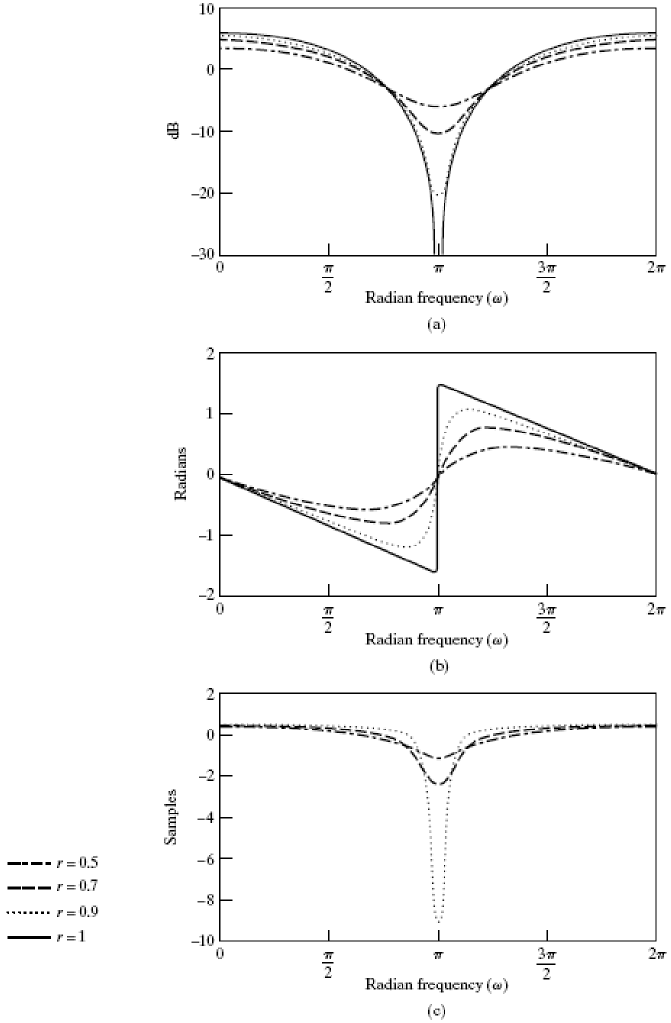


Figure 5.11 Frequency response for a single zero, with $\theta = \pi$, $r = 1$, 0.9, 0.7, and 0.5. (a) Log magnitude. (b) Phase. (c) Group delay for $r = 0.9$, 0.7, and 0.5.

Frequency response of the single zero system by varying the parameter θ.

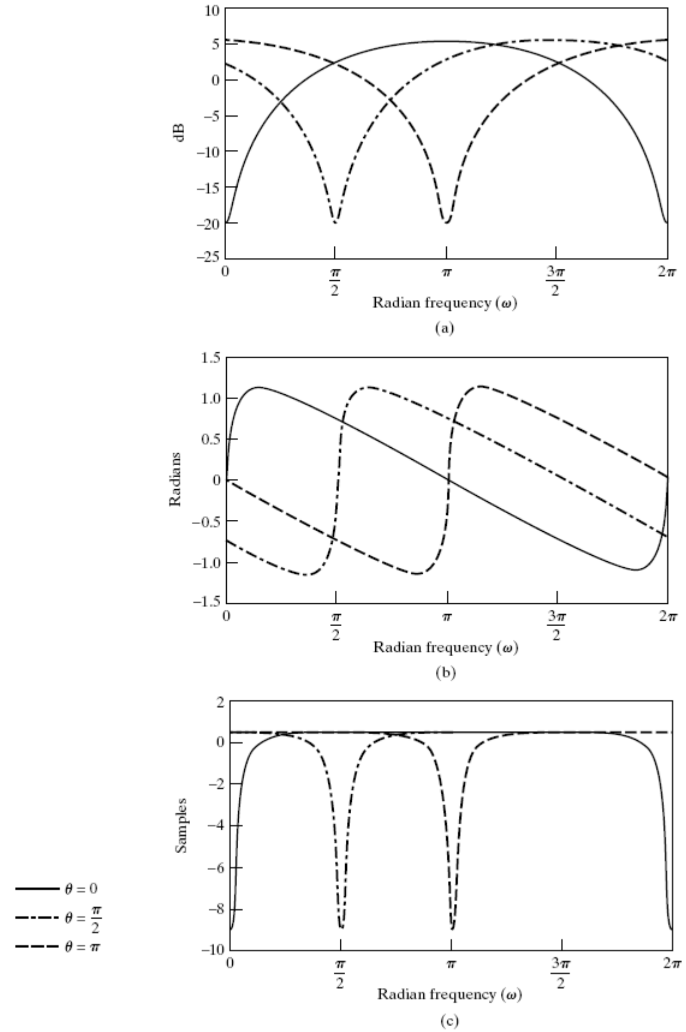
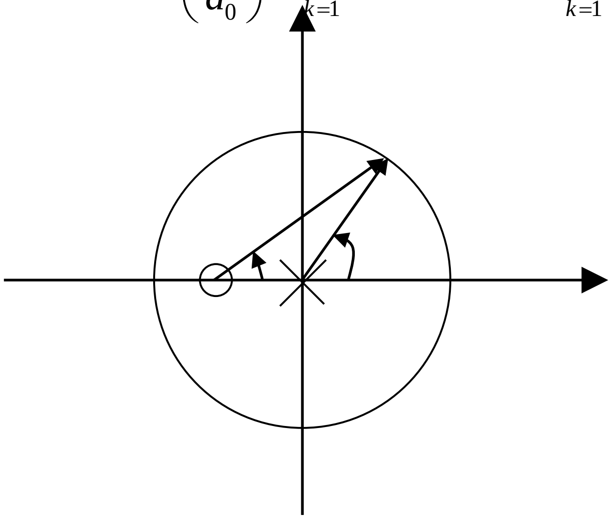


Figure 5.8 Frequency response for a single zero, with $r = 0.9$ and the three values of θ shown. (a) Log magnitude. (b) Phase. (c) Group delay.

Magnitude and Phase: Vector Geometry

$$H(z)=\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}=\left(\frac{b_0}{a_0}\right)z^{N-M}\frac{\prod_{k=1}^{M}\left(z-c_k\right)}{\prod_{k=1}^{N}\left(z-d_k\right)}\quad \text{(assume } N>M\text{)}$$

$$\Rightarrow\begin{cases}\left|H(e^{j\omega})\right|=\left|\dfrac{b_0}{a_0}\right|\dfrac{\prod_{k=1}^{M}|e^{j\omega}-c_k|}{\prod_{k=1}^{N}|e^{j\omega}-d_k|}\\ \angle H(e^{j\omega})=\angle\left(\dfrac{b_0}{a_0}\right)+\sum_{k=1}^{M}\angle(e^{j\omega}-c_k)-\sum_{k=1}^{N}\angle(e^{j\omega}-d_k)+(N-M)\omega\end{cases}$$



As seen in the single-zero/pole example,

• the magnitude of the system can be computed by noting the length of the vectors.

• the phase can be computed by noting the angle of the zero and pole makes with the vector.

Example: Second-Order IIR System (Conjugate Poles)

Given:

$$H(z)=\frac{1}{\left(1-re^{j\theta}z^{-1}\right)\left(1-re^{-j\theta}z^{-1}\right)}=\frac{z^2}{\left(z-re^{j\theta}\right)\left(z-re^{-j\theta}\right)}$$

The magnitude of $H\left(e^{j\omega}\right)$ is

$$\left|H\left(e^{j\omega}\right)\right|=\frac{\left|e^{j2\omega}\right|}{\left|e^{j\omega}-re^{j\theta}\right|\left|e^{j\omega}-re^{-j\theta}\right|}=\frac{|\mathbf{v}_3|}{|\mathbf{v}_1||\mathbf{v}_2|}$$

(see Fig. 5.15). So the log magnitude is

$$\begin{aligned}20\log_{10}\left|H\left(e^{j\omega}\right)\right|&=10\log_{10}\left|H\left(e^{j\omega}\right)\right|^2\\&=10\log_{10}1-10\log_{10}\left|e^{j\omega}-re^{j\theta}\right|^2-10\log_{10}\left|e^{j\omega}-re^{-j\theta}\right|^2\\&=-10\log_{10}\left(1-2r\cos(\omega-\theta)+r^2\right)-10\log_{10}\left(1-2r\cos(\omega+\theta)+r^2\right)\end{aligned}$$

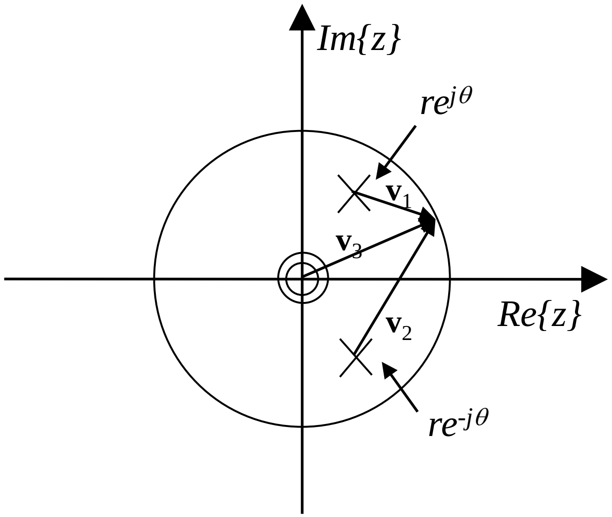


Figure 5.15: Pole-zero plot of two poles example.

The phase is:

$$\begin{aligned}\angle H(e^{j\omega})&=\angle 1-\angle(1-re^{j\theta}e^{-j\omega})-\angle(1-re^{-j\theta}e^{-j\omega})\\&=-\angle\left[1-\left(r\cos\theta+jr\sin\theta\right)\left(\cos\omega-j\sin\omega\right)\right]-\angle\left[1-\left(r\cos\theta-jr\sin\theta\right)\left(\cos\omega-j\sin\omega\right)\right]\\&=-\angle\left[1-\left(r\cos\omega\cos\theta-jr\sin\omega\cos\theta+jr\cos\omega\sin\theta+r\sin\omega\sin\theta\right)\right]\\&\quad-\angle\left[1-\left(r\cos\omega\cos\theta-jr\sin\omega\cos\theta-jr\cos\omega\sin\theta-r\sin\omega\sin\theta\right)\right]\\&=-\angle\left[1-\left(r\cos(\omega-\theta)-jr\sin(\omega-\theta)\right)\right]-\angle\left[1-\left(r\cos(\omega+\theta)-jr\sin(\omega+\theta)\right)\right]\\&=-\angle\left[1-r\cos(\omega-\theta)+jr\sin(\omega-\theta)\right]-\angle\left[1-r\cos(\omega+\theta)+jr\sin(\omega+\theta)\right]\\&=-\tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)-\tan^{-1}\left(\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)}\right)\end{aligned}$$

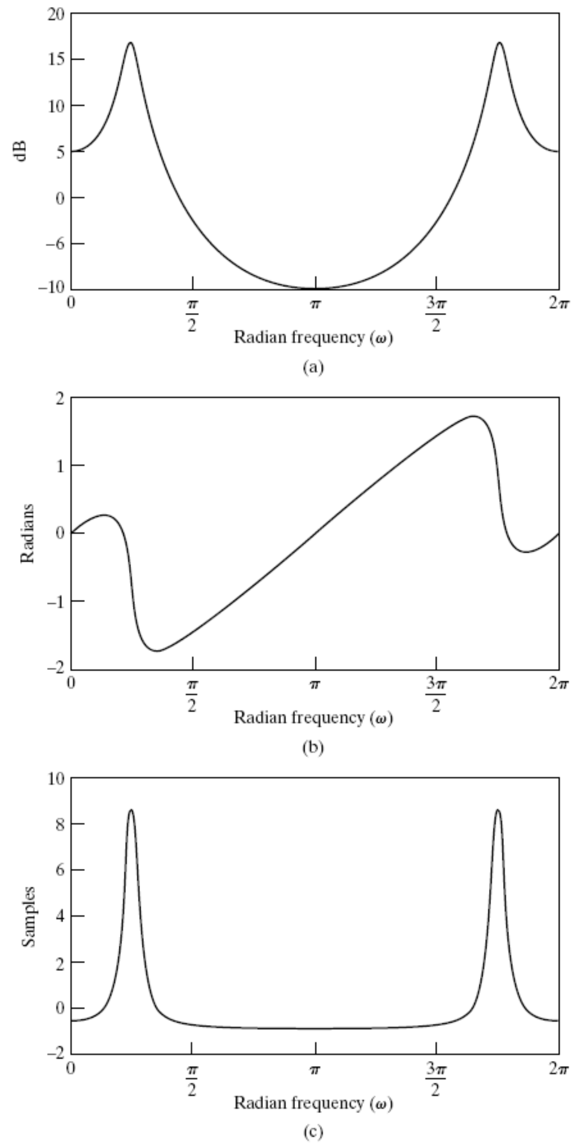


Figure 5.16 Frequency response for a complex-conjugate pair of poles as in Example 5.8, with $r = 0.9$, $\pi/4$. (a) Log magnitude. (b) Phase. (c) Group delay.

Example: Second-Order FIR Systems (Conjugate Poles)

$$H(z) = \left(1 - re^{j\theta}z^{-1}\right)\left(1 - re^{-j\theta}z^{-1}\right) = 1 - 2r\cos\theta z^{-1} + r^2 z^{-2}$$

This is the reciprocal of the second-order pole example. So, the frequency response plots for this FIR system are the negative of the plots in Figure 5.16 above. The pole and zero locations are now interchanged in the reciprocal.

Relationship Between Magnitude and Phase

- For rational system functions, there is some constraint between magnitude and phase
- Given the *number* of poles and zeros, and the *magnitude* (*phase*) response, there are only a *finite* number of possible *phase* (*magnitude*) responses.

E.g. Given magnitude (square), try to decide its phase

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = H(z)H^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j\omega}}$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)} \qquad H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k^* z\right)}$$

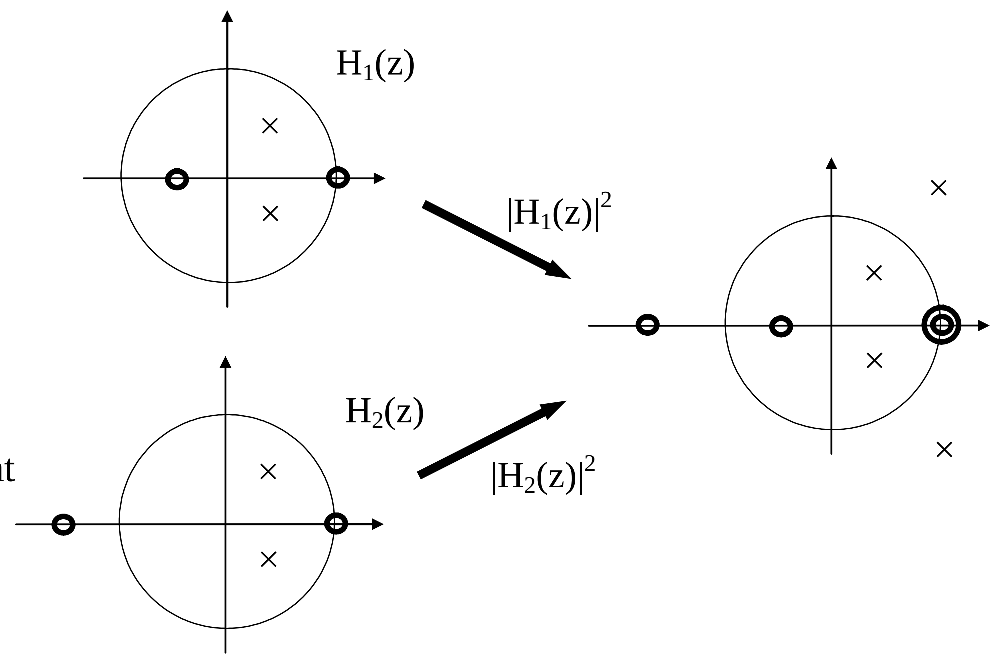
$$\left(\text{because } \left(1\text{-}c_k z^{-1}\right)^* = \left(1-c_k^*\left(z^{-1}\right)^*\right) \Rightarrow \left(1-c_k^* z\right) \text{ for } z \to \frac{1}{z^*}\right)$$

Relationship between Magnitude and Phase

- For each pole d_k of $H(z)$ => poles d_k and $(d_k^*)^{-1}$ of $C(z)$
- *For each zero c_k of $H(z)$ => zeros c_k and $(c_k^*)^{-1}$ of $C(z)$*
- In fact, the poles and zeros of $C(z)$ occur in conjugate reciprocal pairs, with one coming from $H(z)$ and the other from $H^*(1/z^*)$
- If $H(z)$ is causal and stable, then we can deduce that all its poles are inside the unit circle, otherwise we cannot. But even with the causality and stability assumption, the location of the zeros cannot be uniquely determined from the zeros of $C(z)$. For example, given two different causal and stable transfer functions, $H_1(z)$ and $H_2(z)$, they still have the same squared magnitude function $C(z)$ despite the fact that they are both causal and stable. Therefore, given that we have access to only $C(z)$, we need more constraints than causality and stability to uniquely identify the locations of the zeros, e.g. *minimum phase* (more about his later).

Let

$$C(z) = H(z)H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)} \frac{\prod_{k=1}^{M}\left(1-c_k^* z\right)}{\prod_{k=1}^{N}\left(1-d_k^* z\right)}$$



All-pass Systems

In an all-pass system:

- the magnitude equals constant
- all frequency components can pass through (but the phase is not linear)
- poles and zeros form a conjugate reciprocal pair

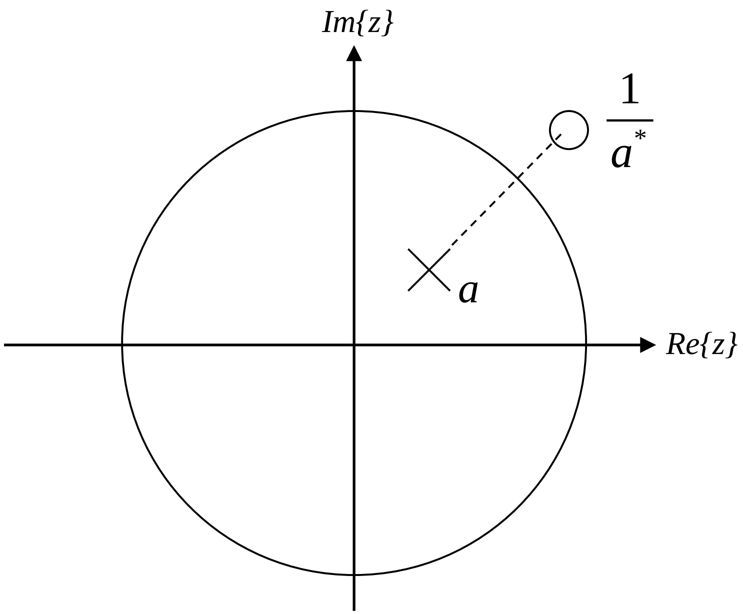
First-order all-pass system:

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

pole: a

zero: $\frac{1}{a^*}$

$$H_{ap}\left(e^{j\omega}\right) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$



Pole-zero diagram of a first-order all-pass system

Example of an All-pass System

Example: $H_{ap}(z) = \dfrac{z^{-1} - 0.9}{1 - 0.9z^{-1}}$ Solid Line

$H_{ap}(z) = \dfrac{z^{-1} + 0.9}{1 + 0.9z^{-1}}$ Dashed Line

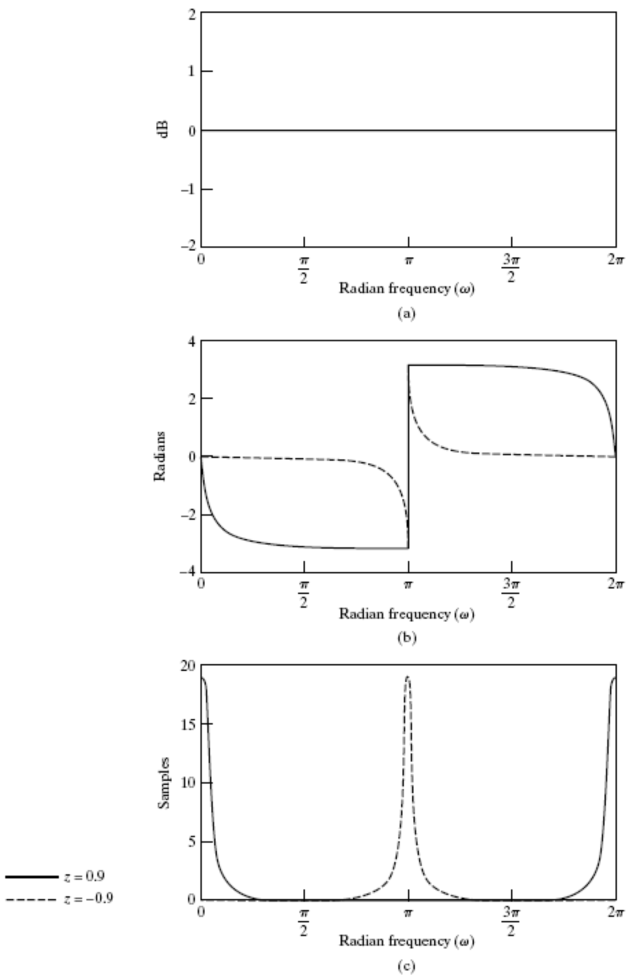


Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

General All-pass Systems

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_k z^{-1}}\prod_{k=1}^{M_c}\frac{\left(z^{-1}-e_k^*\right)\left(z^{-1}-e_k\right)}{\left(1-e_k z^{-1}\right)\left(1-e_k^* z^{-1}\right)} \qquad (5.92)$$

M_r : number of real poles

M_c : number of complex-valued poles

$M = N = 2M_c + M_r$

For $a = re^{j\theta}$, $\begin{cases} |H_{\text{ap}}(e^{j\omega})| = \text{ constant} \\ \angle\left[\dfrac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right] = -\omega - 2\arctan\left[\dfrac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] \end{cases}$

- A general all-pass filter is a product of the first-order and second-order factors.
- The continuous phase of a causal all-pass filter is always non-positive for $0 < \omega < \pi$. This can be proven by showing that the group delay of a first-order all-pass system is non-negative. *(Pf)*

For $a = re^{j\theta}$, $grd\left[\dfrac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right] = \dfrac{1-r^2}{1+r^2-2r\cos\left(\omega-\theta\right)} = \dfrac{1-r^2}{\left|1-re^{j\theta}e^{-j\omega}\right|^2} \geq 0,$

because $r < 1$ (since the system is causal and stable). Therefore, the denominator and numerator are positive, thus the group delay is always positive. Since the group delay of higher order all-pass systems will be a sum of $\dfrac{1-r^2}{\left|1-re^{j\theta}e^{-j\omega}\right|^2}$, therefore, group delay of any all-pass systems will be positive

General All-pass Systems

- Since

$$\arg\left[H_{ap}\left(e^{j\omega}\right)\right] = -\int_0^{\omega} grd\left[H_{ap}\left(e^{j\phi}\right)\right]d\phi + \arg\left[H_{ap}\left(e^{j0}\right)\right], \text{ for } 0 \le \omega \le \pi,$$

  and the phase of the 2nd term is always zero because from (5.92)

$$H_{ap}\left(e^{j0}\right) = A\prod_{k=1}^{M_r}\frac{1-d_k}{1-d_k}\prod_{k=1}^{M_c}\frac{\left|1-e_k\right|^2}{\left|1-e_k\right|^2} = A.$$

  Since grd$[H_{ap}(e^{j\omega})] \ge 0$, therefore $arg[H_{ap}(e^{j\omega})] \le 0$, for $0 \le \omega \le \pi$.

- All-pass filters can be used as phase compensators. They are useful in transforming frequency-selective lowpass filters into other frequency-selective forms and in obtaining variable-cutoff filters.

Minimum-Phase Systems

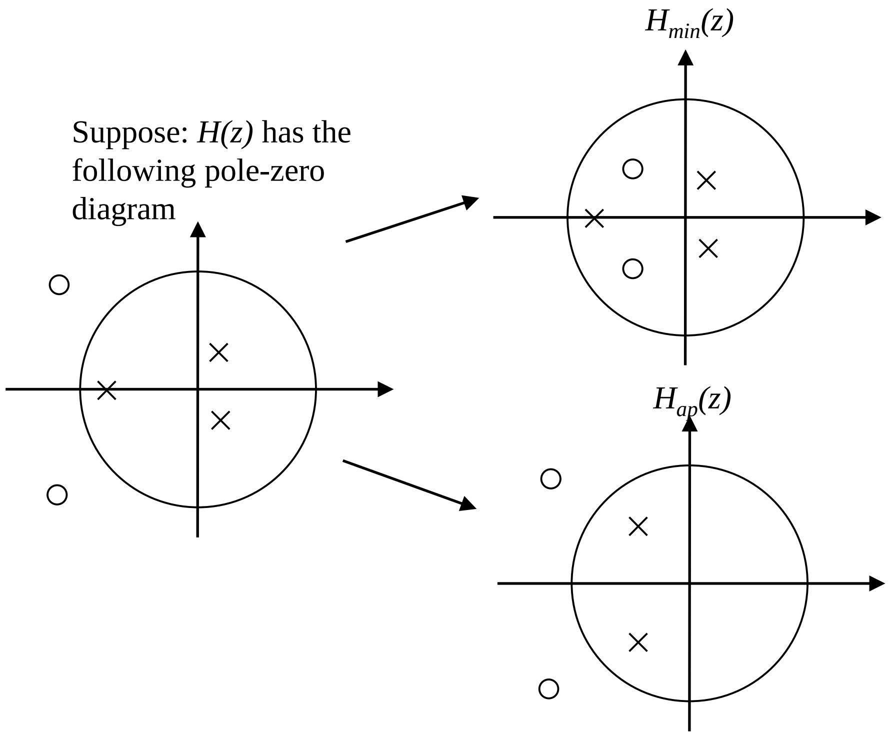
- Magnitude response does not completely characterize the LTI system. If we know the system is causal and stable, then we have only restricted the location of the poles, but not the location of the zeros. However, if we place similar restrictions on the inverse of the system, then the location of the zeros can also be specified.
- For minimum-phase systems, $H(z)$ and its inverse $1/H(z)$ are both causal and stable
  - All poles and zeros are inside the unit circle
- Given the specification on the magnitude squared response, a unique minimum-phase system can be determined

Factorization of Rational System Functions

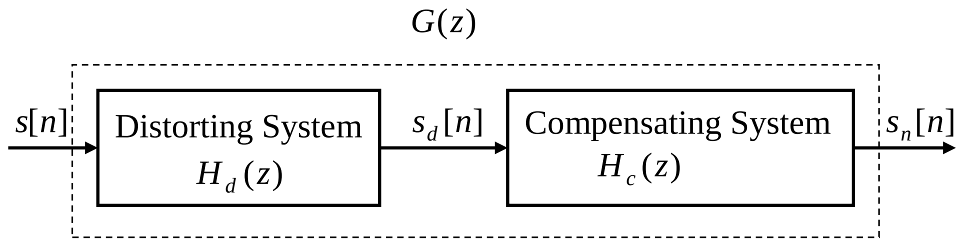
- For any (stable, causal) rational system function $H(z)$, it can be express by $H(z) = H_{min}(z)\, H_{ap}(z)$
  - *Proof*: Suppose $H(z)$ has one zero ($z=1/c^*$, $|c| < 1$) outside the unit circle (and the remaining poles and zeros are inside the unit circle). Then

$$H(z) = H_1(z)\left(z^{-1} - c^*\right)$$
$$= H_1(z)\left(1 - cz^{-1}\right)\frac{z^{-1} - c^*}{1 - cz^{-1}}$$

    where $H_1(z)$ is, by definition, a minimum-phase system. Therefore, $H_1(z)(1-cz^{-1})$ is also minimum phase. The above procedure can be extended to general cases to include more zeros outside the unit circle.
- Note that $H_{min}(z)$ contains the poles and zeros of $H(z)$ that lie inside the unit circle, plus zeros that are the conjugate reciprocals of the zeros of $H(z)$ that lie outside the unit circle. Then $H_{ap}(z)$ contains the zeros of $H(z)$ that are outside the unit circle and the conjugate reciprocal poles inside the unit circle to cancel those zeros from $H_{min}(z)$



Application of MP System – Frequency Response Compensator



Design $H_c(z)$ such that $G(z) = H_d(z)H_c(z)$ is desired.

For example, if we wish $G(z) = H_d(z)H_c(z) = \text{constant}$

Let $H_d(z) = H_{d\min}(z)H_{ap}(z)$, then choose $H_c(z) = \dfrac{1}{H_{d\min}(z)}$ instead.

Then $G(z) = H_{ap}(z) \Rightarrow |G(z)| = 1$, and $\angle G(e^{j\omega}) = \angle H_{ap}(e^{j\omega})$

Frequency Response Compensator

$$H_d(z) = \left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)\underbrace{\left(1-1.25e^{j0.8\pi}z^{-1}\right)\left(1-1.25e^{-j0.8\pi}z^{-1}\right)}_{\text{zeros outside unit circle}}$$

Then we can obtain $H_c(z)$ by obtaining $H_{d\min}(z)$. This is done by reflecting the zeros of $H(z)$ that are outside the unit circle to inside the unit circle. Then $H_{ap}(z)$ is obtained to cancel out the reflected zeros.

$$H_d(z) = H_{d\min}(z)H_{ap}(z)$$
$$= \left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)(1.25)^2\left(z^{-1}-0.8e^{-j0.8\pi}\right)\left(z^{-1}-0.8e^{j0.8\pi}\right)$$

$$\Rightarrow H_{d\min}(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})$$

$$\Rightarrow H_{ap}(z) = \frac{(z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi})}{(1-0.8e^{-j0.8\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})}$$

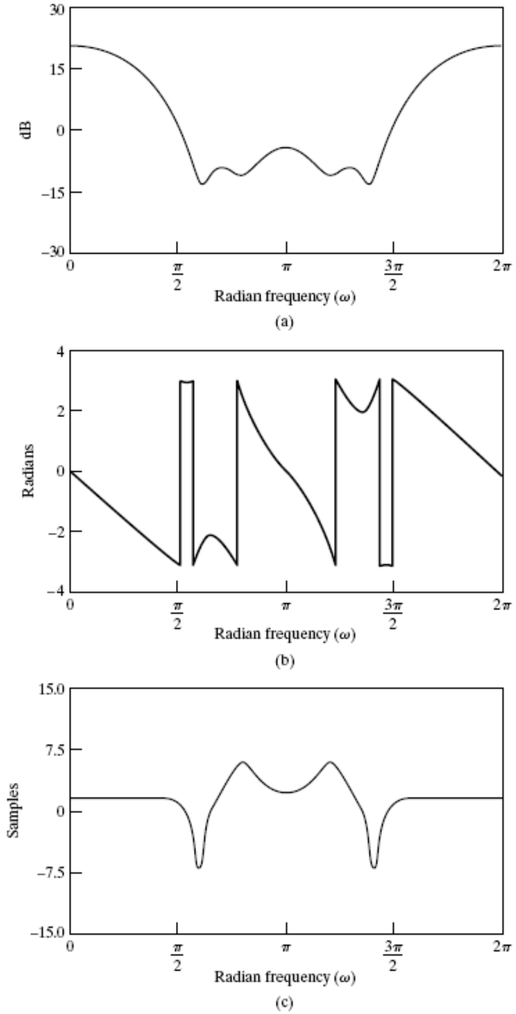


Figure 5.27 Frequency response for FIR system with pole–zero plot in Figure 5.26. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

$H_{min}(z)$

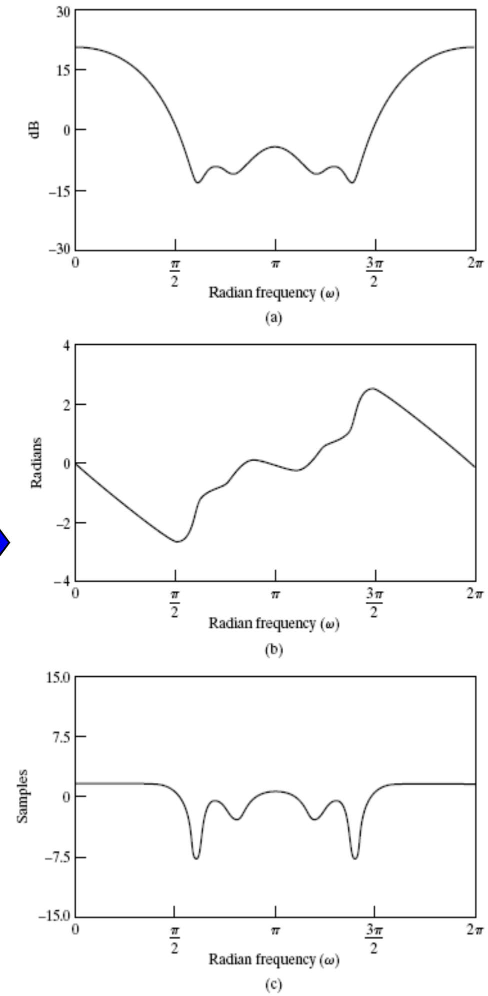


Figure 5.28 Frequency response for minimum-phase system in Example 5.15. (a) Log magnitude. (b) Phase. (c) Group delay.

$H_{ap}(z)$

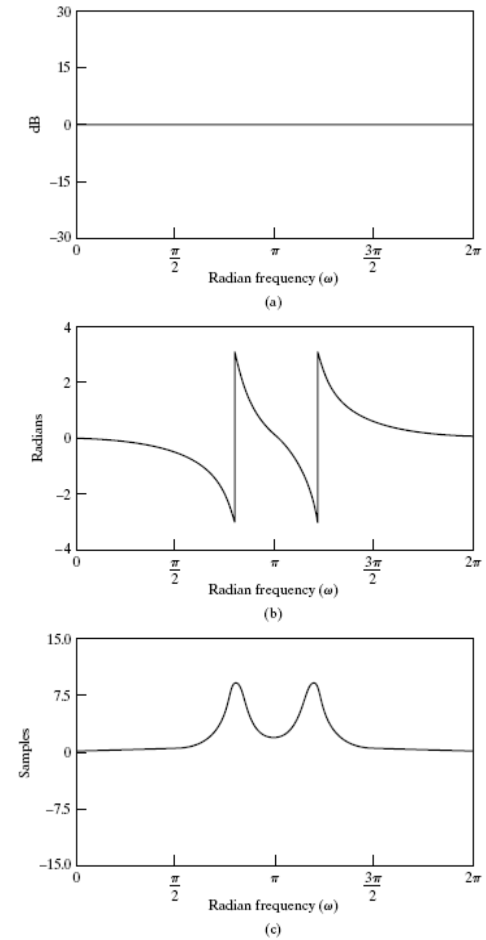


Figure 5.29 Frequency response of all-pass system of Example 5.15. (The sum of corresponding curves in Figures 5.28 and 5.29 equals the corresponding curve in Figure 5.27 with the sum of the phase curves taken modulo 2π.) (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

Properties of Min. Phase Systems

- **Minimum phase-lag property**
  - Why this type of systems is called "minimum phase"?
  - Given the magnitude specification of a system, find the one that has the least phase-lag. ➔ Minimum-phase system.
  - Proof: $H(z) = H_{\min}(z) \cdot H_{ap}(z)$

    $$\Rightarrow \arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

    It was shown before that for allpass filters, $arg[H_{ap}(e^{j\omega})] < 0,\ 0 \le \omega \le \pi$. Thus, $arg[H(e^{j\omega})] = arg[H_{min}(e^{j\omega})]$ + negative value

    Hence, $H_{min}(z)$ has the minimum phase-lag, i.e. the phase of $H_{min}(z)$ is less negative than that of $H(e^{j\omega})$.
- *Remark:* To ensure the minimum phase-lag property, (in addition to the pole and zero locations), we require that $H(e^{j\omega}) > 0$, at $\omega = 0$, i.e. $H(e^{j0}) = \sum_n h[n] > 0$. This is because $h[n]$ and $-h[n]$ both have the same magnitude but the phase will be different by a factor of π radians. So to remove the ambiguity, we must impose this condition

Properties of Min. Phase Systems

- **Minimum group-delay property**

$$grd\left[H\left(e^{j\omega}\right)\right] = grd\left[H_{\min}\left(e^{j\omega}\right)\right] + \underbrace{grd\left[H_{ap}\left(e^{j\omega}\right)\right]}_{>0} \qquad 0 \le \omega \le \pi$$

$$\Rightarrow grd\left[H_{\min}\left(e^{j\omega}\right)\right] < grd\left[H\left(e^{j\omega}\right)\right]$$

Properties of Min. Phase Systems

- **Minimum energy-delay property**
  - The partial energy of a minimum-phase system is most concentrated around $n = 0$, i.e. the energy of a minimum-phase system is delayed the least of all systems having the same magnitude response function. See Fig. 5.32a. Define **partial energy** (of impulse response) to be $E[n] = \sum_{m=0}^{n} |h[m]|^2$
    Then the minimum-phase system $H_{min}(z)$ has the largest $E[n]$ among all possible $H(z)$. That is, it accumulates more energy up to n.

Min. phase

Max. phase

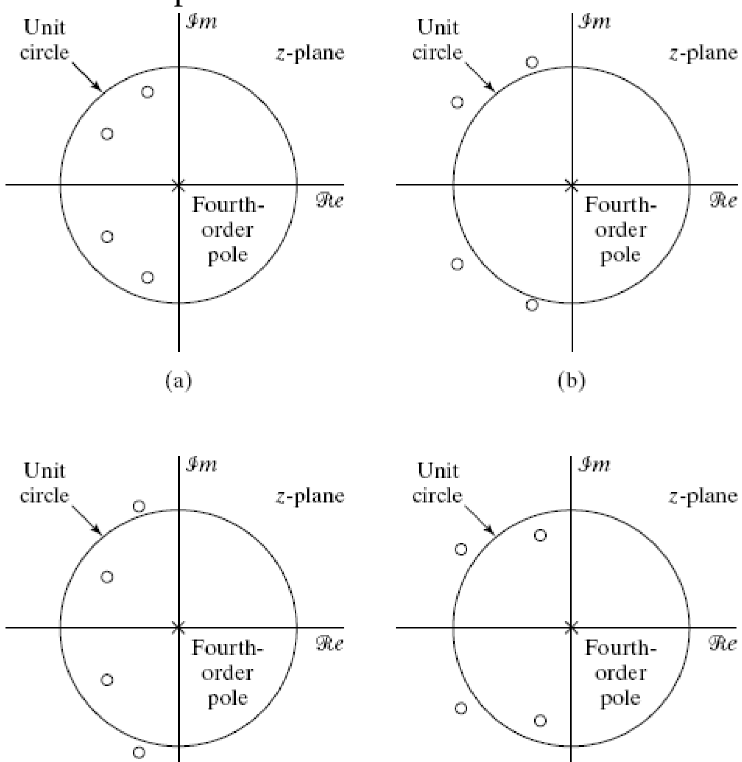
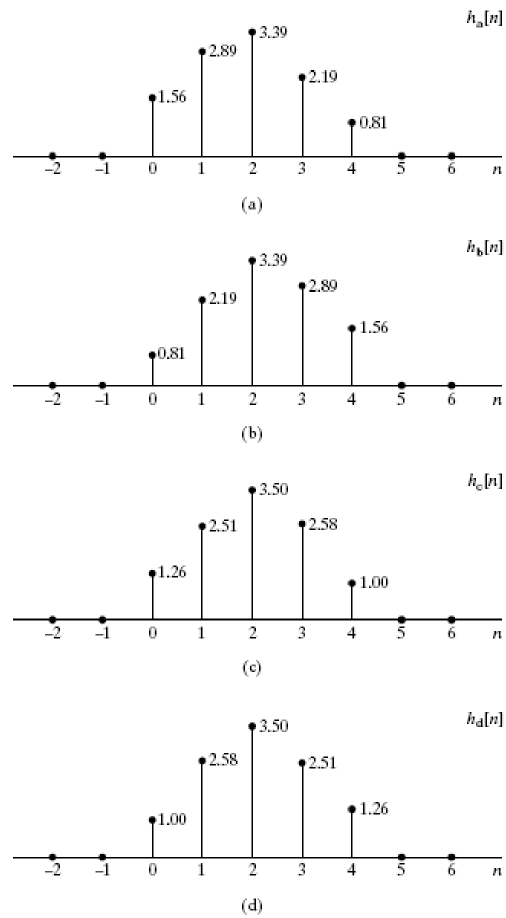


Figure 5.30 Four systems, all having the same frequency-response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals.



Min. phase

Max. phase

Figure 5.31 Sequences corresponding to the pole–zero plots of Figure 5.30.

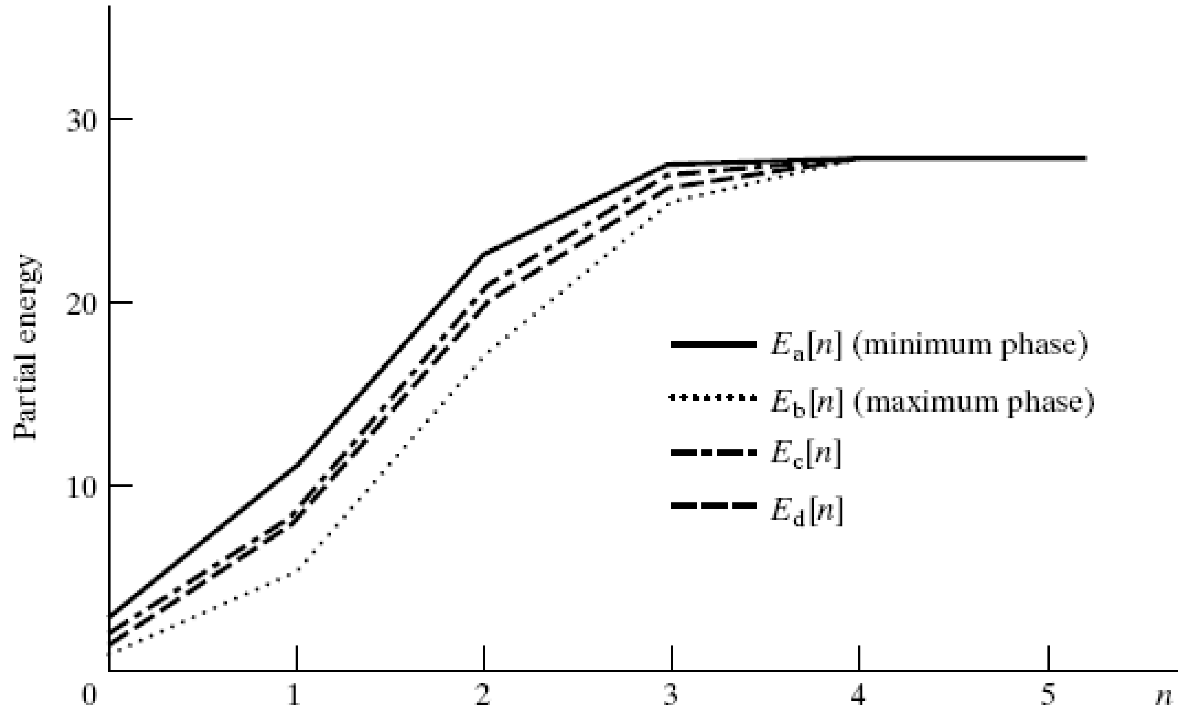


Figure 5.32 Partial energies for the four sequences of Figure 5.31. (Note that $E_a[n]$ is for the minimum-phase sequence $h_a[n]$ and $E_b[n]$ is for the maximum-phase sequence $h_b[n]$.)

Linear Phase Systems

- Zero phase systems are not realizable for real-time systems
  - $h[n] = h[-n], \quad \text{for } n \in \mathbb{Z} \;\Rightarrow\; H\left(e^{j\omega}\right)$ is real and even
- Ideal delay systems: $H_{id}\left(e^{j\omega}\right) = e^{-j\omega\alpha} \;\Leftrightarrow\; h_{id}[n] = \delta[n-\alpha]$

$$\begin{cases} \left|H_{id}\left(e^{j\omega}\right)\right| = 1 \\ \angle H_{id}\left(e^{j\omega}\right) = -\omega\alpha \\ grd\left[H_{id}\left(e^{j\omega}\right)\right] = \alpha \end{cases}$$

- Linear phase: $H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{-j\omega\alpha}, \quad |\omega| < \pi$

$$\begin{cases} \left|H\left(e^{j\omega}\right)\right| = \text{any} \\ \angle H\left(e^{j\omega}\right) = -\omega\alpha, \quad \alpha: \text{ can be non-integer} \end{cases}$$

Example

The impulse response of a causal lowpass filter symmetric about $n = n_d$ is

$$h_{\ell p}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}$$

A zero-phase LPF (i.e. ideal LPF) can be defined as

$$\hat{H}_{\ell p}(e^{j\omega}) = H_{\ell p}(e^{j\omega}) e^{j\omega n_d} = \left| H_{\ell p}(e^{j\omega}) \right|,$$

where the impulse response is shifted left by n_d. Therefore,

the causal LPF does not have zero phase (phase $= -\omega n_d$)

$$H_{\ell p}(e^{j\omega}) = \left| H_{\ell p}(e^{j\omega}) \right| e^{-j\omega n_d}$$

Linear Phase Systems and Its Relationship to the Impulse Response

E.g. Symmetry on $h[n]$ in the ideal delay system:

$$\text{Let} \quad H_{lp}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c \\ 0, & \text{otherwise,} \end{cases}$$

$$\Leftrightarrow \; h_{lp}[n] = \frac{\sin\left[\omega_c(n-\alpha)\right]}{\pi(n-\alpha)}$$

1. $\alpha = \text{integer}$

   $h_{lp}[n]$ is an even function $\left(h[2\alpha - n] = h[n]\right)$

α = point of symmetry

2. $\alpha = \text{integer} + \dfrac{1}{2}$

   $h_{lp}[n]$ is an even function $\left(h[2\alpha - n] = h[n]\right)$

3. Otherwise,

   $h_{lp}[n]$ has no symmetry

- For case 2 and 3, even though the delay is not an integer, but we can interpret the output using the idea of continuous time processing of discrete-time signals, i.e. the continuous-time filter, $h_c(t)$, is equal to $h_c(t) = \delta(t - \alpha T)$ or $H_c(j\Omega) = e^{-j\Omega\alpha T}$. So that the frequency response of the effective (discrete-time) system is $H_{eff}\left(e^{j\omega}\right) = e^{-j\omega\alpha}, \; |\omega| < \pi$
- From case 1 and 2, they suggest that symmetry (in these two cases, even symmetry) is sufficient to guarantee linear phase

Linear Phase Systems and Its Relationship to the Impulse Response

- Observations
  - In case 1 and 2, the signal is symmetric, i.e. $h[2\alpha-n] = h[n]$.
  - In case 1, since α=5 is an integer, we can shift the impulse response to the left by α to obtain a signal that has zero-phase. This is not true for case 2 since α = 4.5 is not an integer
  - The symmetric property in cases 1 and 2 is only sufficient, and not necessary, to obtain linear phase. This is so because the impulse response in Figure 5.35c also has linear phase but it is not symmetric (the group delay is a constant α = 4.3)

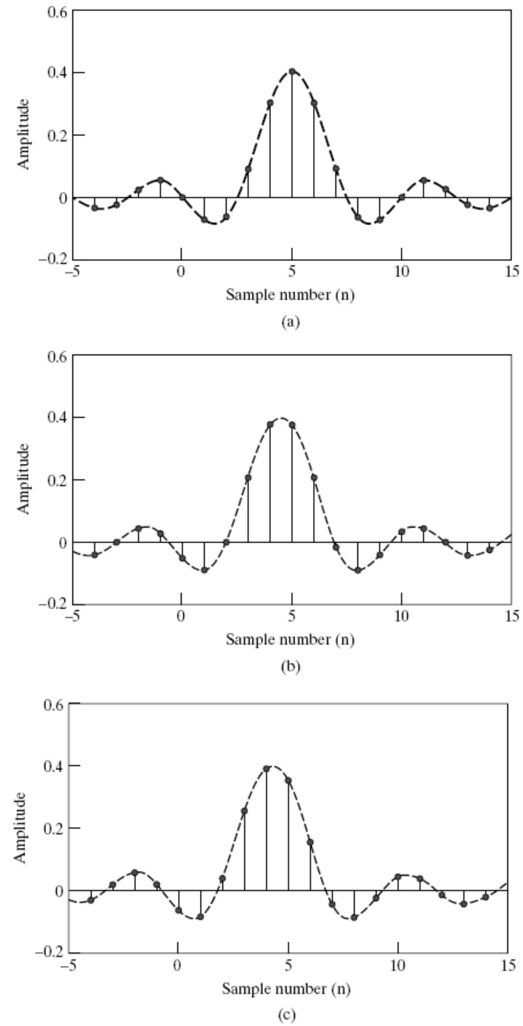


Figure 5.35 Ideal lowpass filter impulse responses, with $\omega_c = 0.4\pi$. (a) Delay = α = 5. (b) Delay = α = 4.5. (c) Delay = α = 4.3.

Generalized Linear Phase

$$H\left(e^{j\omega}\right) = A\left(e^{j\omega}\right) \cdot e^{-j\alpha\omega + j\beta}, \quad \alpha, \beta\text{: real constants}$$

$A(e^{j\omega})$ is called the amplitude response. It is real.

Special case: $\beta = 0 \quad \rightarrow$ linear phase

$$\begin{cases} \text{Group delay:} & \tau(\omega) = \text{grd}\left[H(e^{j\omega})\right] = \alpha \\ \text{Phase:} & \arg\left[H(e^{j\omega})\right] = \beta - \alpha\omega, \quad 0 < \omega < \pi \end{cases}$$

Essentially, this system has a *constant* group delay

Symmetry Property of Generalized Linear Phase Systems

We can observe the symmetry by decomposing $H\left(e^{j\omega}\right)$ as follows:

$$H\left(e^{j\omega}\right) = A\left(e^{j\omega}\right)e^{j(\beta-\alpha\omega)}$$
$$= A\left(e^{j\omega}\right)\cos\left(\beta-\alpha\omega\right) + jA\left(e^{j\omega}\right)\sin\left(\beta-\alpha\omega\right)$$

$H\left(e^{j\omega}\right)$ is also equal to:

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n]\cos\omega n - j\sum_{n=-\infty}^{\infty} h[n]\sin\omega n$$

Since tangent of phase = $\mathrm{Im}\{\bullet\} / \mathrm{Re}\{\bullet\}$, and using the two equations above (assuming that $h[n]$ is real), we have

$$\tan\left(\beta-\alpha\omega\right) = \frac{\sin\left(\beta-\alpha\omega\right)}{\cos\left(\beta-\alpha\omega\right)} = \frac{-\sum_{n=-\infty}^{\infty} h[n]\sin\omega n}{\sum_{n=-\infty}^{\infty} h[n]\cos\omega n}$$

Note that $A\left(e^{j\omega}\right)$ has disappeared.

Symmetry Property of Generalized Linear Phase Systems

Cross-multiplying the above, we have

$$\sin(\beta - \alpha\omega) \sum_{n=-\infty}^{\infty} h[n] \cos \omega n = -\cos(\beta - \alpha\omega) \sum_{n=-\infty}^{\infty} h[n] \sin \omega n$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} h[n] \left\{ \cos \omega n \sin(\beta - \alpha\omega) + \sin \omega n \cos(\beta - \alpha\omega) \right\} = 0$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} h[n] \sin\left[\omega(n-\alpha) + \beta\right] = 0, \quad \forall \omega, \quad \text{eqn. (*)}$$

This is a necessary condition on $h[n]$, α, and β for the system to have constant group delay.

However, there are many solutions that satisfy eqn.(*).

Solution 1

$$\sum_{n=-\infty}^{\infty} h[n]\sin\left[\omega(n-\alpha)+\beta\right]=0, \quad \forall\omega, \quad \text{eqn. (*)}$$

M is the **order** of the filter $\longrightarrow$ $\begin{cases} \beta = 0 \text{ or } \pi \\ 2\alpha = M = \text{ integer} \\ h[2\alpha - n] = h[n] \quad \text{(even symmetry)} \end{cases}$

To see if this is really a solution to (*), we plug it into (*). Let $\beta = 0$. We get:

$$\sum_{n=-\infty}^{\infty} h[n]\sin\omega\left(n-\frac{M}{2}\right)$$

$$=h\left[\frac{M}{2}\right]\sin\omega\left(\frac{M}{2}-\frac{M}{2}\right)+\sum_{n=-\infty}^{\frac{M}{2}-1} h[n]\sin\omega\left(n-\frac{M}{2}\right)+\sum_{n=\frac{M}{2}+1}^{\infty} h[n]\sin\omega\left(n-\frac{M}{2}\right)$$

(From the 2^{nd} term, let $p = M - n$)

$$=h\left[\frac{M}{2}\right]\sin\omega\left(\frac{M}{2}-\frac{M}{2}\right)+\sum_{p=\frac{M}{2}+1}^{\infty} h[p]\sin\omega\left(M-p-\frac{M}{2}\right)+\sum_{n=\frac{M}{2}+1}^{\infty} h[n]\sin\omega\left(n-\frac{M}{2}\right)$$

$$=h\left[\frac{M}{2}\right]\sin\omega\left(\frac{M}{2}-\frac{M}{2}\right)+\sum_{n=\frac{M}{2}+1}^{\infty} h[n]\left[\sin\omega\left(n-\frac{M}{2}\right)+\sin\omega\left(M-n-\frac{M}{2}\right)\right]$$

$$=\sum_{n=\frac{M}{2}+1}^{\infty} h[n]\left[\sin\omega\left(n-\frac{M}{2}\right)+\sin\omega\left(\frac{M}{2}-n\right)\right]$$

$$=0 \quad \left(\text{matches with result: } \sum_{n=-\infty}^{\infty} h[n]\sin\left[\omega(n-\alpha)+\beta\right]=0\right)$$

Note: We only require that $h[n]$ satisfies a symmetric property. There is no constraint on the numerical values of $h[n]$. *Remark*: It will be shown later that $A\left(e^{j\omega}\right)$ is even (and real).

Solution 2

$$\sum_{n=-\infty}^{\infty} h[n]\sin\left[\omega(n-\alpha)+\beta\right]=0,\quad \forall\omega,\quad \text{eqn. (*)}$$

$$\begin{cases} \beta = \pi/2 \text{ or } 3\pi/2 \\ 2\alpha = M = \text{ integer} \quad \overset{\overbrace{\beta=\frac{\pi}{2}}}{\Rightarrow} \quad \sin\left[\omega\left(n-\dfrac{M}{2}\right)+\dfrac{\pi}{2}\right]=\cos\omega\left(n-\dfrac{M}{2}\right) \\ h[2\alpha-n]=-h[n] \quad \text{(odd symmetry) (implies the point of symmetry = 0)} \end{cases}$$

To see if this is really a solution to (*), we plug it into (*). Let $\beta=\pi/2$. We get:

$$\sum_{n=-\infty}^{\infty} h[n]\cos\omega\left(n-\frac{M}{2}\right)$$

$$=\sum_{n=-\infty}^{\frac{M}{2}-1} h[n]\cos\omega\left(n-\frac{M}{2}\right)+\sum_{n=\frac{M}{2}+1}^{\infty} h[n]\cos\omega\left(n-\frac{M}{2}\right)$$

$$=-\sum_{p=\frac{M}{2}+1}^{\infty} h[p]\cos\omega\left(M-p-\frac{M}{2}\right)+\sum_{n=\frac{M}{2}+1}^{\infty} h[n]\cos\omega\left(n-\frac{M}{2}\right) \qquad \text{(From the 1}^{st}\text{ term, let } p=M-n\text{)}$$

$$=\sum_{n=\frac{M}{2}+1}^{\infty} h[n]\left[\cos\omega\left(n-\frac{M}{2}\right)-\cos\omega\left(\frac{M}{2}-n\right)\right]$$

$$=0 \quad \text{(matches with result: } \sum_{n=-\infty}^{\infty} h[n]\sin\left[\omega(n-\alpha)+\beta\right]=0\text{)}$$

Remark: It will be shown later that $A\left(e^{j\omega}\right)$ is odd (and real).

There are other possible solutions, for example, fractional delay.

FIR Systems With Generalized Linear Phase: Even Symmetry

- Previously, we've shown that (even) symmetry in $h[n]$ is sufficient for the system to have generalized linear phase
- If a generalized linear-phase system is also causal, then (*) becomes

$$\sum_{n=0}^{\infty} h[n]\sin(\omega n-\omega\alpha+\beta)=0, \quad \forall\omega$$

Under this causal condition and the even symmetry condition, i.e. $h[2\alpha-n]=h[n]$,

$$h[n]=0, \quad n<0 \text{ and } n>M.$$

This implies that $h[n]$ is an FIR filter.

More precisely, if the filter length is $M+1$, and

$$h[n]=\begin{cases} h[M-n], & 0\le n\le M, \\ 0, & otherwise \end{cases} \quad \text{(symmetric w.r.t. } M/2\text{)}$$

then, $H(e^{j\omega})=A_e(e^{j\omega})e^{-j\omega M/2}$, where $A_e(e^{j\omega})$ is real and even

FIR Systems With Generalized Linear Phase: Odd Symmetry

If

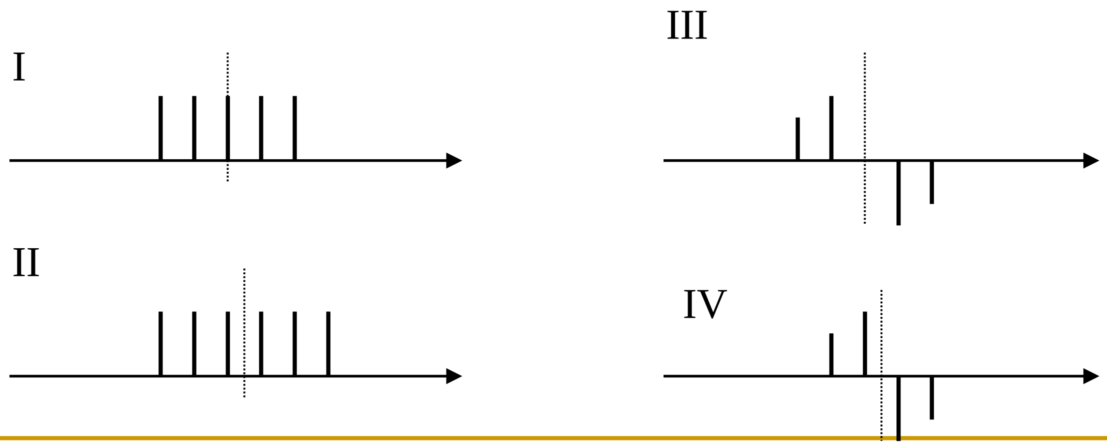
$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M, \\ 0, & otherwise \end{cases} \quad \text{(anti-symmetric w.r.t. } M/2\text{)}$$

then, $H\left(e^{j\omega}\right) = jA_o\left(e^{j\omega}\right)e^{-j\omega M/2} = A_o\left(e^{j\omega}\right)e^{-j\omega M/2 + j\pi/2}$, where $A_o\left(e^{j\omega}\right)$ is real and odd

Remark : Nearly all linear phase filters are FIR filters. There are special types of IIR filters that have linear phase, but they cannot be implemented by difference equations. The above two cases are the most common ones

Four Types of Linear Phase FIR Filters

$$
\begin{cases}
\text{Type-I.} & h[n] = h[M-n] \\
 & M \text{ even } \left(\text{or } \dfrac{M}{2}, \text{ an integer}\right) \\
\text{Type-II.} & h[n] = h[M-n] \\
 & M \text{ odd } \left(\text{or } \dfrac{M}{2}, \text{ a half integer}\right) \\
\text{Type-III.} & h[n] = -h[M-n] \\
 & M \text{ even} \\
\text{Type-IV.} & h[n] = -h[M-n] \\
 & M \text{ odd}
\end{cases}
$$



Type-I Linear Phase FIR Filter

We can show Type-I FIR's have linear-phase by checking its Fourier Transform.

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$=\sum_{n=0}^{M/2-1} h[n]e^{-j\omega n}+h[M/2]e^{-j\omega M/2}+\sum_{n=M/2+1}^{M} h[n]e^{-j\omega n}$$

$$=\sum_{n=0}^{M/2-1} h[n]e^{-j\omega n}+h[M/2]e^{-j\omega M/2}+\sum_{k=0}^{M/2-1} h[M-k]e^{-j\omega(M-k)} \qquad \left(\text{let } n=M-k \ \Rightarrow\ k=M-n\right)$$

$$=\sum_{n=0}^{M/2-1} h[n]e^{-j\omega n}+h[M/2]e^{-j\omega M/2}+\sum_{k=0}^{M/2-1} h[k]e^{-j\omega(M-k)} \quad \left(\text{for Type-I filter: } h[k]=h[M-k]\right)$$

$$=\sum_{n=0}^{M/2-1} h[n]\left(e^{-j\omega n}+e^{-j\omega(M-n)}\right)+h[M/2]e^{-j\omega M/2}$$

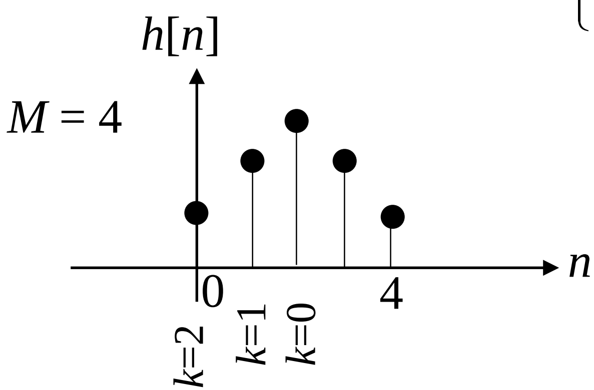
$$=\sum_{n=0}^{M/2-1} h[n]e^{-\frac{j\omega M}{2}}\left(e^{-j\omega n}e^{\frac{j\omega M}{2}}+e^{j\omega n}e^{-\frac{j\omega M}{2}}\right)+h[M/2]e^{-j\omega M/2}$$

$$=e^{-j\omega M/2}\left\{\sum_{n=0}^{M/2-1} 2h[n]\cos\left[\omega\left(n-\frac{M}{2}\right)\right]+h[M/2]\right\}$$

Type-I Linear Phase FIR Filter

Let $k = -\left(n - \frac{M}{2}\right) \Rightarrow n = -k + \frac{M}{2}$. Then $n = 0 \Rightarrow k = \frac{M}{2}$, $n = \frac{M}{2} - 1 \Rightarrow k = -\frac{M}{2} + 1 + \frac{M}{2} = 1$

$$H\left(e^{j\omega}\right) = e^{-j\omega M/2} \sum_{k=0}^{M/2} a[k]\cos(\omega k), \qquad \text{where } a[k] = \begin{cases} h\left[\frac{M}{2}\right], & k = 0, \\ 2h[n], & k = 1, \ldots, \frac{M}{2} \end{cases}$$



$$k = 0 \Rightarrow n = \frac{M}{2} = 2,$$

$$k = 1 \Rightarrow n = -1 + \frac{M}{2} = 1,$$

$$k = \frac{M}{2} \Rightarrow n = 0$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contribute a phase of 0 or π (in case $h[n]$ is negative) to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega M/2 \ \text{ or } \ -\omega M/2 + \pi.$$

The phase of $H(e^{j\omega})$ is linear by definition of linear phase $-j\alpha+\beta$, where

$$\alpha = M/2, \quad \beta = 0 \text{ or } \pi.$$

Type-II Linear Phase FIR Filter

We can show Type-II FIR's have linear-phase by checking its Fourier Transform.

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{M}h[n]e^{-j\omega n}$$

$$=\sum_{n=0}^{(M-1)/2}h[n]e^{-j\omega n}+\sum_{n=(M+1)/2}^{M}h[n]e^{-j\omega n}$$

$$=\sum_{n=0}^{(M-1)/2}h[n]e^{-j\omega n}+\sum_{k=0}^{(M-1)/2}h[M-k]e^{-j\omega(M-k)} \qquad \left(\text{let } n=M-k \;\Rightarrow\; k=M-n\right)$$

$$=\sum_{n=0}^{(M-1)/2}h[n]e^{-j\omega n}+\sum_{k=0}^{(M-1)/2}h[k]e^{-j\omega(M-k)} \qquad \left(\text{for Type-II filter: } h[k]=h[M-k]\right)$$

$$=\sum_{n=0}^{(M-1)/2}h[n]\left(e^{-j\omega n}+e^{-j\omega(M-n)}\right)$$

$$=e^{-j\omega M/2}\sum_{n=0}^{(M-1)/2}2h[n]\cos\left[\omega\left(n-\frac{M}{2}\right)\right]$$

Type-II Linear Phase FIR Filter

$$\text{Let } k=\frac{M}{2}-n-\frac{1}{2} \quad \Rightarrow \quad n=\frac{M-1}{2}-k, \quad n=0\Rightarrow k=\frac{M-1}{2}, \quad n=\frac{M-1}{2}\Rightarrow k=0$$

$$H\left(e^{j\omega}\right)=e^{-j\omega M/2}\sum_{k=0}^{(M-1)/2} 2h\left[\frac{M-1}{2}-k\right]\cos\left[\omega\left(-k-\frac{1}{2}\right)\right]$$

$$=e^{-j\omega M/2}\sum_{k=0}^{(M-1)/2} 2h\left[\frac{M-1}{2}-k\right]\cos\left[\omega\left(k+\frac{1}{2}\right)\right]$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since h[n] is real, the second term in the product above contributes a phase of 0 or π to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega M/2 \quad \text{or} \quad -\omega M/2+\pi.$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = M/2, \quad \beta = 0 \text{ or } \pi.$$

Type-III Linear Phase FIR Filter

For Type-III FIR linear-phase systems, note that at n = M/2,

h[M/2] = -h[M-(M/2)] = -h[M/2],

so h[M/2] = 0.

We can show Type-III FIR's have linear-phase by checking its Fourier Transform.

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$=\sum_{n=0}^{M/2-1} h[n]e^{-j\omega n}+h[M/2]e^{-j\omega M/2}+\sum_{n=M/2+1}^{M} h[n]e^{-j\omega n}$$

$$=\sum_{n=0}^{M/2-1} h[n]e^{-j\omega n}+\sum_{k=0}^{M/2-1} h[M-k]e^{-j\omega(M-k)} \qquad \left(\begin{array}{l}\text{for Type-III filter: } h[M/2]=0.\\ \text{Let } n=M-k \;\Rightarrow\; k=M-n\end{array}\right)$$

$$=\sum_{n=0}^{M/2-1} h[n]e^{-j\omega n}-\sum_{k=0}^{M/2-1} h[k]e^{-j\omega(M-k)} \qquad \left(\text{for Type-III filter: } h[k]=-h[M-k]\right)$$

$$=\sum_{n=0}^{M/2-1} h[n]\left(e^{-j\omega n}-e^{-j\omega(M-n)}\right)=e^{-j\omega M/2}(-j)\sum_{n=0}^{M/2-1} 2h[n]\sin\left[\omega\left(n-\frac{M}{2}\right)\right]$$

Type-III Linear Phase FIR Filter

$$\text{Let } k = \frac{M}{2} - n \quad \Rightarrow \quad n = \frac{M}{2} - k, \quad n = 0 \Rightarrow k = \frac{M}{2}, \quad n = \frac{M}{2} - 1 \Rightarrow k = 1$$

$$H\left(e^{j\omega}\right) = e^{-j\omega M/2}\left(-j\right)\sum_{k=1}^{M/2} 2h\left[\frac{M}{2} - k\right]\sin\left[\omega\left(\frac{M}{2} - k - \frac{M}{2}\right)\right]$$

$$= e^{-j\omega M/2}\left(-j\right)\sum_{k=1}^{M/2} 2h\left[\frac{M}{2} - k\right]\sin\left[\omega\left(-k\right)\right]$$

$$= e^{-j\omega M/2} j\sum_{k=1}^{M/2} 2h\left[\frac{M}{2} - k\right]\sin\left(\omega k\right)$$

The first term $je^{-j\omega M/2}$ gives a phase of $-\omega M/2+\pi/2$ to $H(e^{j\omega})$. Since h[n] is real, the second term in the product above contribute a phase of 0 or π to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega M/2 + \pi/2 \quad \text{or} \quad -\omega M/2 + 3\pi/2.$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = M/2 \ , \quad \beta = \pi/2 \ \text{ or } \ 3\pi/2.$$

Type-IV Linear Phase FIR Filter

We can show Type-IV FIR's have linear-phase by checking its Fourier Transform.

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{k=0}^{(M-1)/2} h[M-k]e^{-j\omega(M-k)} \qquad (\text{let } n = M-k \ \Rightarrow\ k = M-n) \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} - \sum_{k=0}^{(M-1)/2} h[k]e^{-j\omega(M-k)} \qquad (\text{for Type-IV filter: } h[k] = -h[M-k]) \\
&= \sum_{n=0}^{(M-1)/2} h[n]\left(e^{-j\omega n} - e^{-j\omega(M-n)}\right) \\
&= e^{-j\omega M/2}(-j)\sum_{n=0}^{(M-1)/2} 2h[n]\sin\left[\omega\left(n-\frac{M}{2}\right)\right]
\end{aligned}
$$

Type-IV Linear Phase FIR Filter

$$\text{Let } k = \frac{M}{2} - n - \frac{1}{2} \quad \Rightarrow \quad n = \frac{M-1}{2} - k, \quad n = 0 \Rightarrow k = \frac{M-1}{2}, \quad n = \frac{M-1}{2} \Rightarrow k = 0$$

$$H\left(e^{j\omega}\right) = e^{-j\omega M/2}(-j)\sum_{k=0}^{(M-1)/2} 2h\left[\frac{M-1}{2} - k\right]\sin\left[\omega\left(\frac{M-1}{2} - k - \frac{M}{2}\right)\right]$$

$$= e^{-j\omega M/2}(-j)\sum_{k=0}^{(M-1)/2} 2h\left[\frac{M-1}{2} - k\right]\sin\left[\omega\left(-k - \frac{1}{2}\right)\right]$$

$$= e^{-j\omega M/2}\, j\sum_{k=0}^{(M-1)/2} 2h\left[\frac{M-1}{2} - k\right]\sin\left[\omega\left(k + \frac{1}{2}\right)\right]$$

The first term $je^{-j\omega M/2}$ gives a phase of $-\omega M/2+\pi/2$ to $H(e^{j\omega})$. Since h[n] is real, the second term in the product above contribute a phase of 0 or π to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega M/2 + \pi/2 \quad \text{or} \quad -\omega M/2 + 3\pi/2.$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = M/2 \;, \quad \beta = \pi/2 \text{ or } 3\pi/2.$$

Type-I Example

$$h[n]=\begin{cases}1 & 0\le n\le 4,\\ 0 & \text{otherwise}\end{cases}\qquad (M=4)$$

$$\Rightarrow\ H\left(e^{j\omega}\right)=e^{-j2\omega}\frac{\sin\left(\frac{5\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

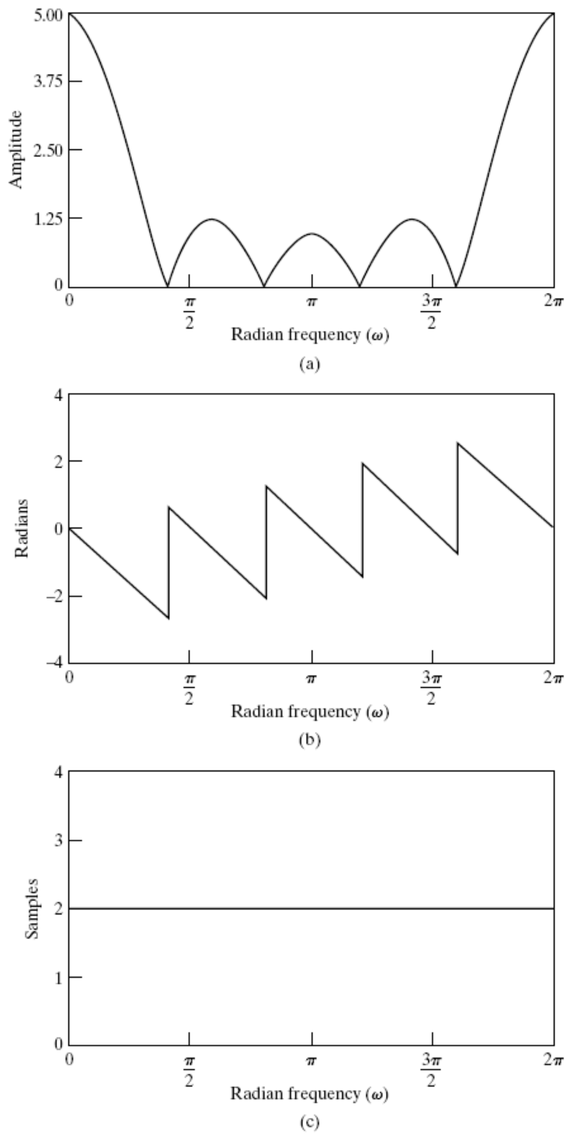


Figure 5.37 Frequency response of type I system of Example 5.17. (a) Magnitude. (b) Phase. (c) Group delay.

Type-II Example

$$h[n]=\begin{cases}1 & 0\le n\le 5,\\ 0 & \text{otherwise}\end{cases}\qquad (M=5)$$

$$\Rightarrow H\left(e^{j\omega}\right)=e^{-j\frac{5}{2}\omega}\frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)}$$

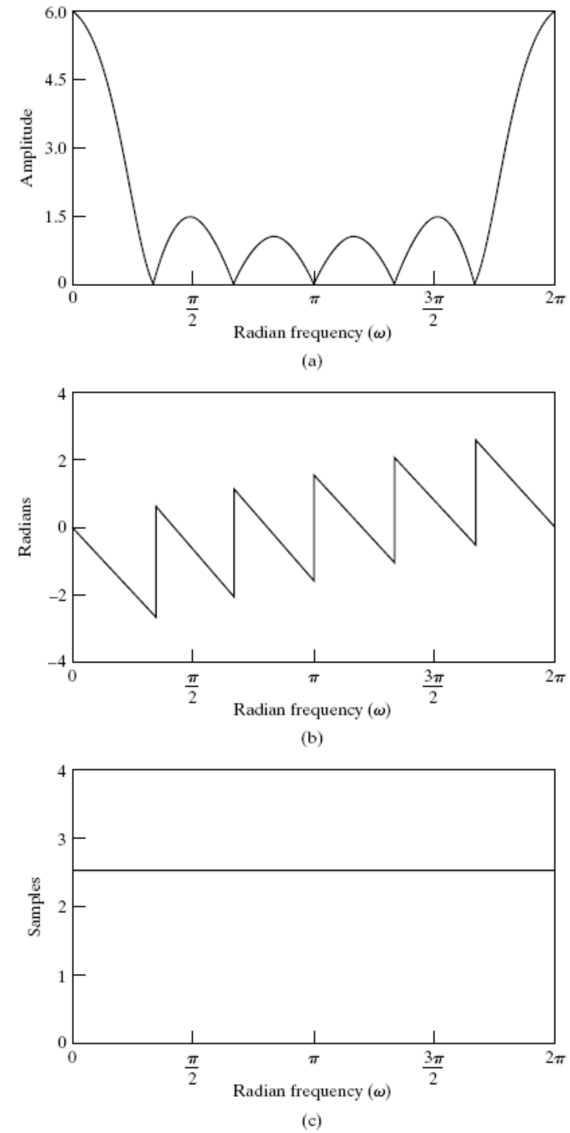


Figure 5.38 Frequency response of type II system of Example 5.18. (a) Magnitude. (b) Phase. (c) Group delay.

Type-III Example

$$h[n] = \delta[n] - \delta[n-2] \qquad (M = 2)$$

$$\Rightarrow H\left(e^{j\omega}\right) = 1 - e^{-j2\omega}$$

$$= j\left(2\sin\omega\right)e^{-j\omega}$$

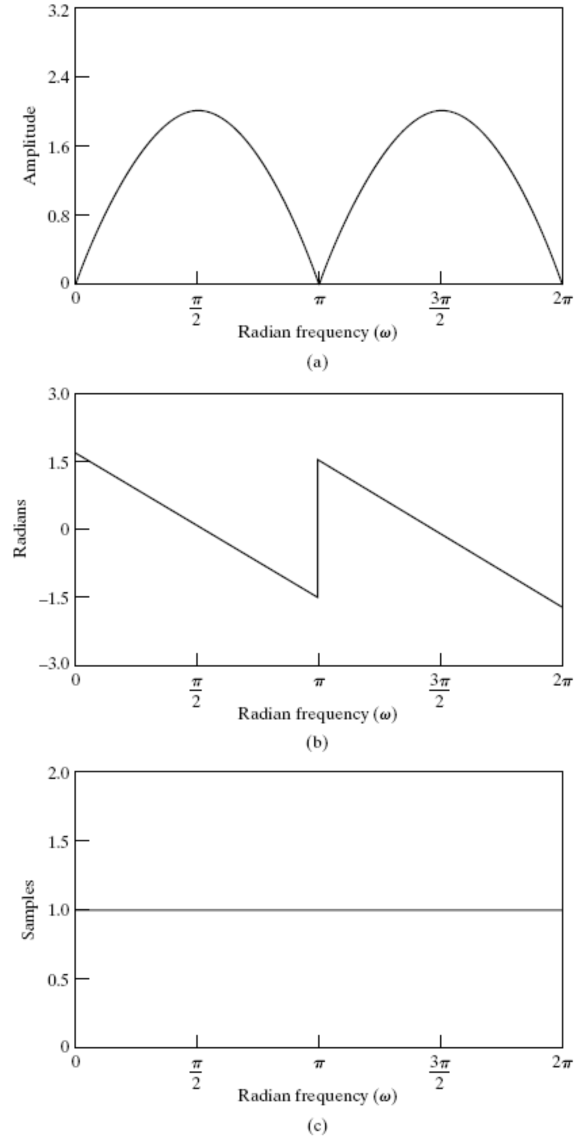


Figure 5.39 Frequency response of type III system of Example 5.19. (a) Magnitude. (b) Phase. (c) Group delay.

Type-IV Example

$$h[n] = \delta[n] - \delta[n-1] \qquad\qquad (M = 1)$$

$$\Rightarrow H\left(e^{j\omega}\right) = 1 - e^{-j\omega}$$

$$= j\left(2\sin\frac{\omega}{2}\right)e^{-j\frac{\omega}{2}}$$

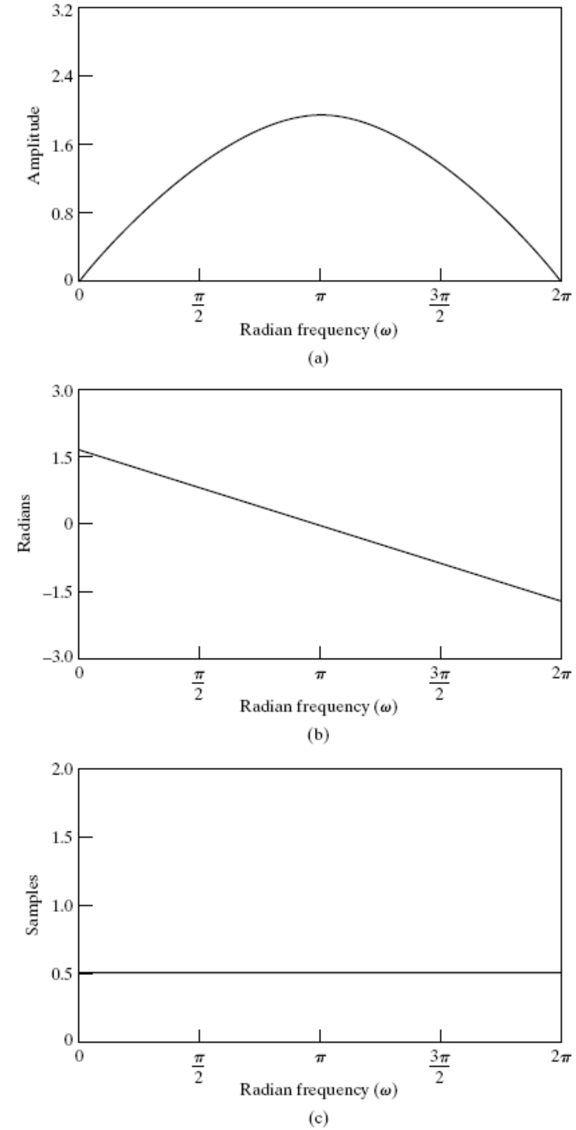


Figure 5.40 Frequency response of type IV system of Example 5.20. (a) Magnitude. (b) Phase. (c) Group delay.

Zeros of Linear Phase FIR Systems

- Causal, stable, FIR systems → no nonzero poles
- Zeros are conjugate pairs (because *h*[*n*] is real)
- Zeros are reciprocal pairs (z_0, z_0^{-1})
  - Proof (Type-I and II systems)

$$
\begin{aligned}
H(z) &= \sum_{n=0}^{M} h[n] \cdot z^{-n} = \sum_{n=0}^{M} h[M-n] \cdot z^{-n} \\
&= \sum_{k=M}^{0} h[k] \cdot z^{k} z^{-M} \qquad (k = M - n) \\
&= z^{-M} H\left(z^{-1}\right)
\end{aligned}
$$

If z_0 is a zero, i.e. $H(z_0) = 0$, then $z_0^{-M} H(z_0^{-1}) = 0$. That is, its reciprocal is a zero too

  - Proof (Type III and IV systems)

$$H(z) = -z^{-M} H\left(z^{-1}\right)$$

If z_0 is a zero, so is its reciprocal

Zeros of Linear Phase FIR Systems

- Type-II (symmetry, odd order) → zero at z = -1
- Type-III (anti-symmetry, even order) → zero at z = 1 and -1
- Type-IV (anti-symmetry, odd order) → zeros at z = 1
  - Proof (Type-I and II systems)

$$H(z) = z^{-M} H\left(z^{-1}\right)$$

$$\text{Let } z = -1, \text{ then } H(-1) = (-1)^{-M} H(-1)$$

$$\text{For } M \text{ even (Type-I), the equality is satisfied}$$

$$\text{For } M \text{ odd (Type-II), } H(-1) \text{ must be } 0 \text{ for the equality to be satisfied}$$

  - Proof (Type III and IV systems)

$$H(z) = -z^{-M} H\left(z^{-1}\right)$$

$$\text{Let } z = 1, \text{ then } H(1) = -H(1)$$

$$\Rightarrow H(1) \text{ must be } 0 \text{ for the equality to be satisfied}$$

$$\text{Let } z = -1, \text{ then } H(-1) = -(-1)^{M} H(-1)$$

$$\Rightarrow \text{For } M \text{ odd (Type-IV), the equality is satisfied}$$

$$\Rightarrow \text{For } M \text{ even (Type-III), } H(-1) \text{ must be } 0 \text{ for the equality to be satisfied}$$

- These constraints on the zeros are useful in designing linear phase FIR systems. For example, for highpass systems, M cannot be odd (zero should not be at z = -1)

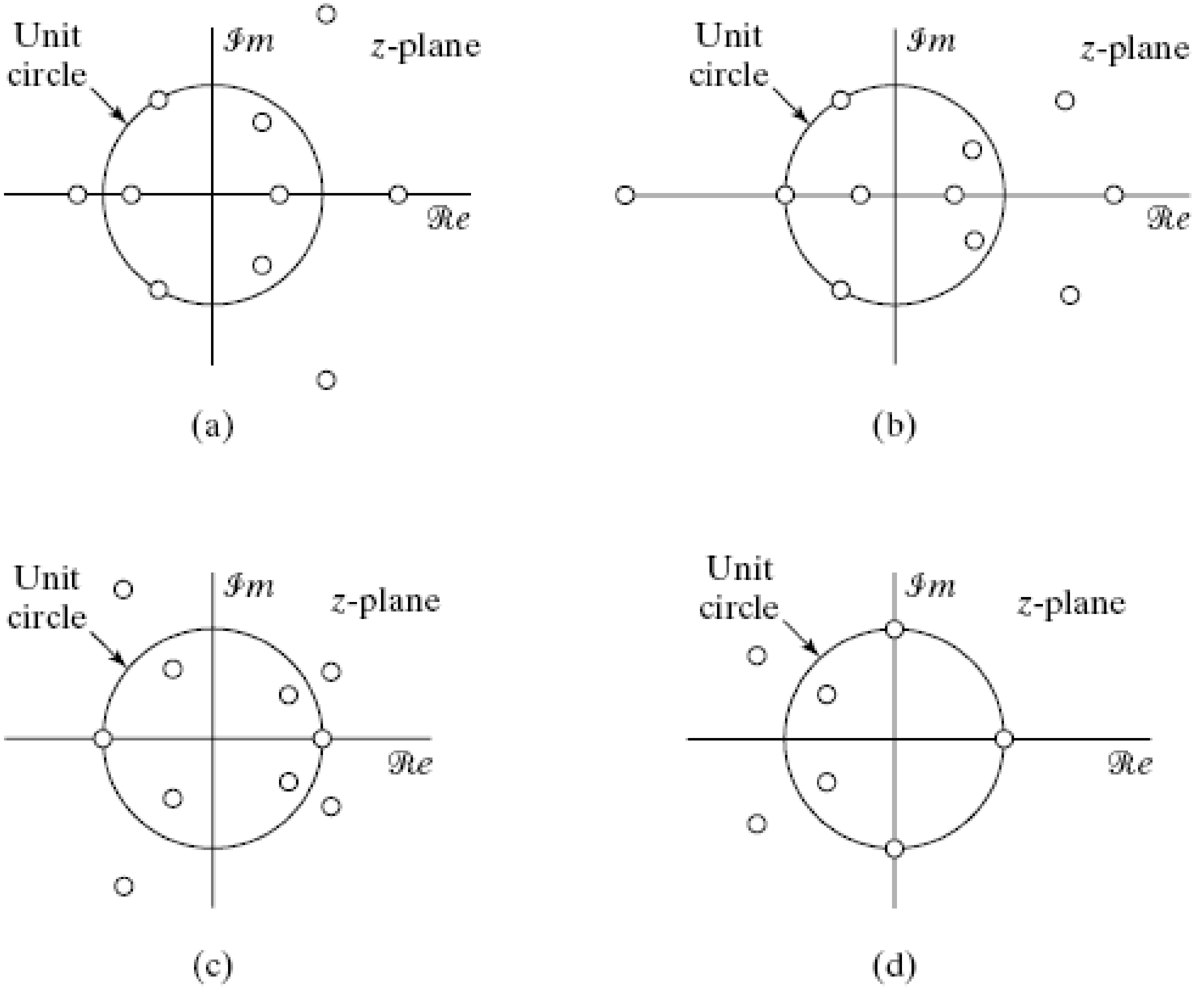


Figure 5.41 Typical plots of zeros for linear-phase systems. (a) Type I. (b) Type II. (c) Type III. (d) Type IV.

TABLE 2.4.1 Four types of real coefficient linear phase FIR filters.

Here $H(z) = \sum_{n=0}^{N} h(n)z^{-n}$, with $h(n)$ real

Type	1	2	3	4
Symmetry	$h(n) = h(N-n)$	$h(n) = h(N-n)$	$h(n) = -h(N-n)$	$h(n) = -h(N-n)$
Parity of N	N even	N odd	N even	N odd
Expression for frequency response $H(e^{j\omega})$	$e^{-j\omega N/2}H_R(\omega)$	$e^{-j\omega N/2}H_R(\omega)$	$je^{-j\omega N/2}H_R(\omega)$	$je^{-j\omega N/2}H_R(\omega)$
Amplitude response or zero-phase response $H_R(\omega)$	$\sum_{n=0}^{M} b_n \cos(\omega n)$ $M = N/2$	$\cos\frac{\omega}{2} \sum_{n=0}^{M} b_n \cos(\omega n)$ $M = (N-1)/2$	$\sin\omega \sum_{n=0}^{M} b_n \cos(\omega n)$ $M = (N-2)/2$	$\sin\frac{\omega}{2} \sum_{n=0}^{M} b_n \cos(\omega n)$ $M = (N-1)/2$
Special features		Zero at $\omega = \pi$	Zero at $\omega = 0$ and π	Zero at $\omega = 0$
Can be used for	Any type of bandpass response (LPF, HPF, etc.)	Any bandpass response except Highpass	Differentiators and and Hilbert transformers†	Differentiators, Hilbert transformers, and high pass filters

$N = M$ in notes

Factorization of Linear Phase Systems

Any linear phase FIR systems can be expressed as

$$\underbrace{H(z)}_{\substack{\text{linear}\\ \text{phase}}} = \underbrace{H_{\min}(z)}_{\substack{\text{min.}\\ \text{phase}}} \cdot \underbrace{H_{\text{uc}}(z)}_{\substack{\text{unit}\\ \text{circle}}} \cdot \underbrace{H_{\max}(z)}_{\substack{\text{max.}\\ \text{phase}}}$$

Because zeros are reciprocals, $H_{\max}(z) = z^{-M} H_{\min}(z^{-1})$, where M_i is the number of zeros of $H_{\min}(z)$ (outside the unit circle). $H_{uc}(z)$ has M_o number of zeros. $H_{\max}(z)$ has M_i number of zeros outside the unit circle

Factorization Example

Given:

$$H_{\min}(z) = (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1})$$

Then

$$H_{\max}(z) = (0.9)^2(1-1.1111e^{j0.6\pi}z^{-1})(1-1.1111e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1})$$

Then

$H(z) = H_{\min}(z)H_{\max}(z)$ has linear phase

$$\Rightarrow \quad 20\log_{10}\left|H(e^{j\omega})\right| = 20\log_{10}\left|H_{\min}(e^{j\omega})\right| + 20\log_{10}\left|H_{\max}(e^{j\omega})\right| = 40\log_{10}\left|H_{\min}(e^{j\omega})\right|$$

$$\Rightarrow \quad \angle H(e^{jw}) = \angle H_{\min}(e^{j\omega}) + \angle H_{\max}(e^{j\omega}) = \angle H_{\min}(e^{j\omega}) + (-\omega M_i - \angle H_{\min}(e^{j\omega})) = -\omega M_i$$

$H(z)$

$H_{min}(z)$

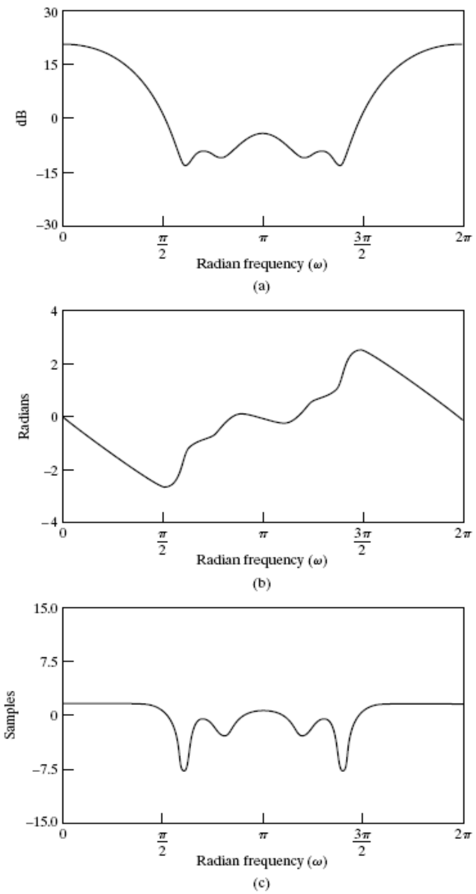


Figure 5.28 Frequency response for minimum-phase system in Example 5.15. (a) Log magnitude. (b) Phase. (c) Group delay.

$H_{max}(z)$

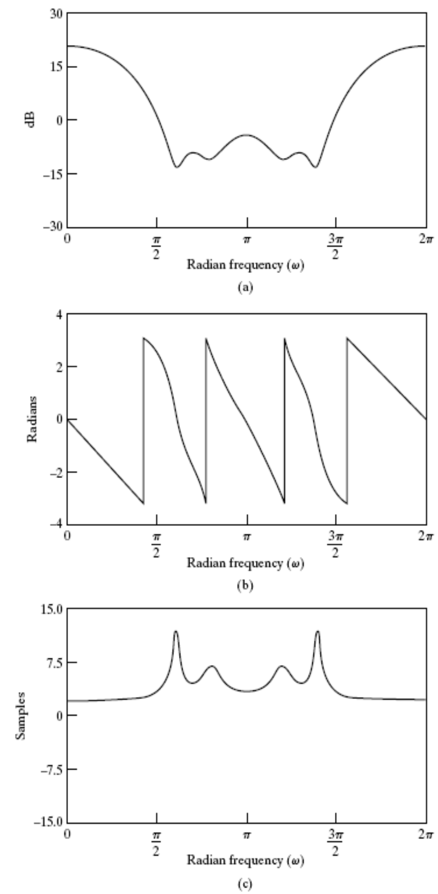


Figure 5.42 Frequency response of maximum-phase system having the same magnitude as the system in Figure 5.28. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

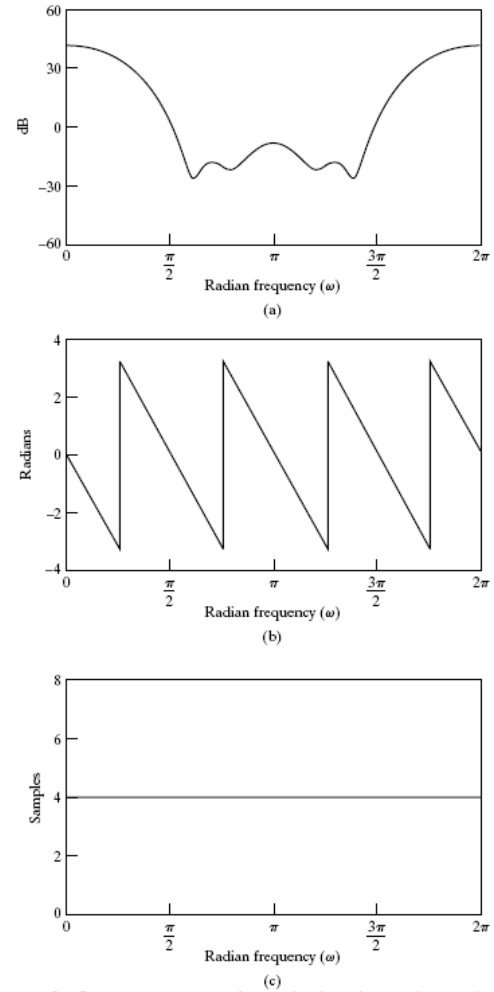


Figure 5.43 Frequency response of cascade of maximum-phase and minimum-phase systems, yielding a linear-phase system. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.