Discrete Fourier Transform (DFT)

Carrson C. Fung
Institute of Electronics
National Yang Ming Chiao Tung University

Introduction

- The DTFT is a theoretical tool to evaluate the frequency response of signals and systems
  - Unfortunately, it cannot be computed using a digital computer
  - Solution: Sample the frequency spectrum ➔ DFT
  - Unlike the CTFT, the DTFT does not have a duality relationship, i.e. if $x(t) \Leftrightarrow X(j\Omega)$, then $X(t) \Leftrightarrow 2\pi x(-j\Omega)$, but no such relationship exists between $x[n]$ and $X(e^{j\omega})$
    - As seen in the sequel, there is a duality relationship between $x[n]$ and $X[k]$
- Idea:
  - Sample the (continuous) function $X(e^{j\omega})$
    - Corresponding function in time now also becomes periodic (but still discrete-time (this is the DFS)
  - Crop out one period of the sequence in time and frequency domain ➔ DFT
    - However, there is an *inherit periodicity* in the time and frequency signals (more later)

Discrete-Fourier Series (DFS)

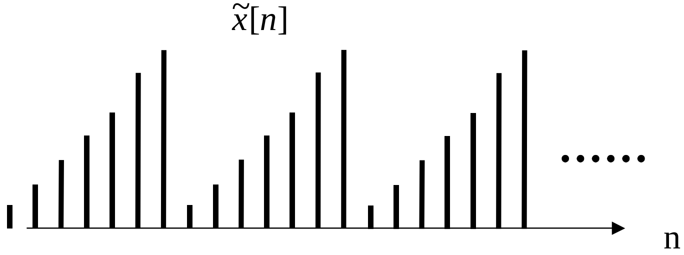
Define: $W_N \triangleq e^{-j2\pi/N}$, thus $W_N^k = e^{-j\frac{2\pi}{N}k}$

Properties of W_N :

- W_N is periodic with period N (it is essentially $\cos$ and $\sin$): $W_N^k = W_N^{k\pm N} = W_N^{k\pm 2N} = \cdots$
- Since W_N is periodic with period N, unlike the CTFS, the DFS representation of a periodic signal $\tilde{x}[n]$ needs only N complex exponentials, i.e.

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

where $\tilde{X}[k]$ is the DFS coefficients of $\tilde{x}[n]$.



$$\tilde{x}[n] = \tilde{x}[n + rN], \qquad period \;\; N$$

Discrete-Fourier Series (DFS)

The DFS representation of $\tilde{x}[n]$ thus becomes

Synthesis equation:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$

Analysis equation:

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}$$

Note: The tilde in $\tilde{x}$ indicates a period signal.

$\tilde{X}[k]$ is periodic with period N.

Is this relationship true?

To show that the analysis and synthesis equations are indeed true, we can derive the analysis equation from the synthesis equation. But first, we need to show that the following property is true:

$$\sum_{k=0}^{N-1} W_N^{\ell k} = \begin{cases} N, & \text{if } \ell = mN \\ 0, & \text{if } \ell \neq mN \end{cases}$$

(Pf) (i) If $\ell = mN$, $W_N^{\ell k} = W_N^{mkN} = 1$. So

$$\sum_{k=0}^{N-1} W_N^{lk} = \sum_{k=0}^{N-1} 1 = N$$

(ii) If $\ell \neq mN$, $W_N^{\ell} \neq 1$, which can be shown by using the geometric series

relation: $\sum_{n=0}^{k} r^n = \frac{1-r^{k+1}}{1-r}$. Then

$$\sum_{k=0}^{N-1} W_N^{\ell k} = \frac{1-W_N^{\ell N}}{1-W_N^{\ell}} = \frac{1-1}{1-W_N^{\ell}} = 0$$

Note that this property can be written more compactly as

$$Y[\ell] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{\ell k} = \sum_{m=-\infty}^{\infty} \delta[\ell - mN]$$

How come the DFS relationship is true?

$$(Pf)\quad \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$

Pick an r where $0\le r\le N-1$. Multiply both sides of the synthesis equation by W_N^{rn} and sum from $n=0$ to $N-1$

$$\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{rn}=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}W_N^{rn}$$

$$=\sum_{k=0}^{N-1}\tilde{X}[k]\left(\frac{1}{N}\sum_{n=0}^{N-1}W_N^{(r-k)n}\right)$$

Now using the property from the last page, we know that

$$\frac{1}{N}\sum_{n=0}^{N-1}W_N^{(r-k)n}=\begin{cases}1, & k-r=mN\\ 0, & else\end{cases}$$

So,

$$\sum_{k=0}^{N-1}\tilde{X}[k]\left(\frac{1}{N}\sum_{n=0}^{N-1}W_N^{(r-k)n}\right)=\sum_{k=0}^{N-1}\tilde{X}[k]\quad (\text{for } r-k=mN)$$

$$=\tilde{X}[0]0+\tilde{X}[1]0+\cdots+\tilde{X}[k=r]1+\cdots$$

$$=\tilde{X}[r]$$

That is, $\tilde{X}[r]=\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{rn}$ QED

Example: Periodic Rectangular Pulse Train

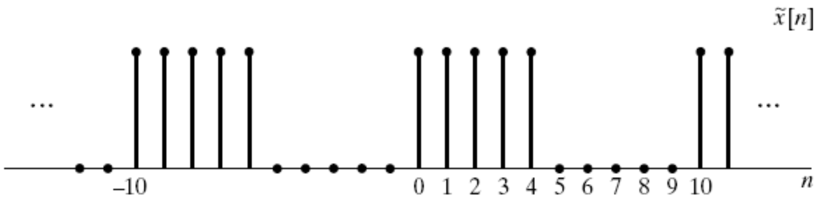


Figure 8.1 Periodic sequence with period $N = 10$ for which the Fourier series representation is to be computed.

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j\frac{4\pi k}{10}} \frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{10}\right)}$$

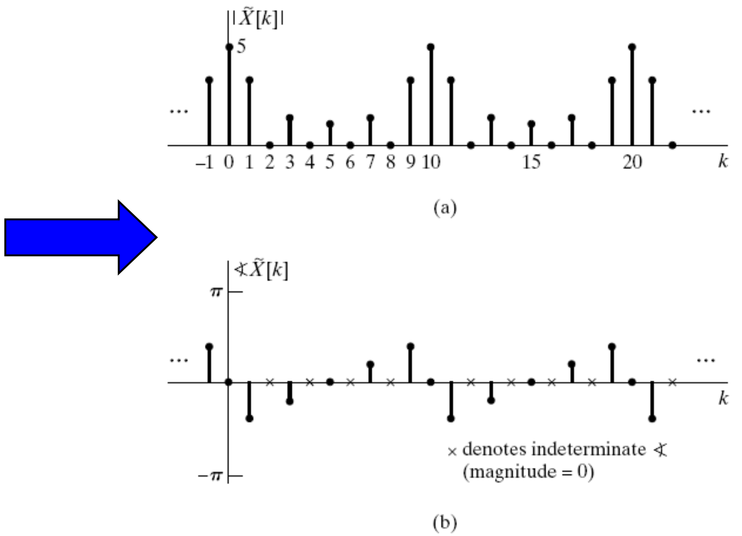


Figure 8.2 Magnitude and phase of the Fourier series coefficients of the sequence of Figure 8.1.

Properties of the DFS

Linearity:

$$\left.\begin{array}{l}\tilde{x}_1[n] \leftrightarrow \tilde{X}_1[k] \\ \tilde{x}_2[n] \leftrightarrow \tilde{X}_2[k]\end{array}\right\}$$

$$\text{then } a\tilde{x}_1[n] + b\tilde{x}_2[n] \leftrightarrow a\tilde{X}_1[k] + b\tilde{X}_2[k]$$

Time Shift:

$$\text{If } \tilde{x}[n] \leftrightarrow \tilde{X}[k]$$

$$\text{then } \tilde{x}[n-m] \leftrightarrow W_N^{km}\tilde{X}[k]$$

Note that any shifts that are greater than or equal to the period, i.e. $m \geq N$ cannot be distinguished in the time domain from a shorter shift m_1 such that $m = m_1 + m_2 N$, where m_1 and m_2 are integers and $0 \leq m_1 \leq N-1$. Or simply, $m_1 = m$ modulo N i.e. m_1 is the remainder of m/N. So $W_N^{km} = W_N^{km_1}$.

Since the DFS coefficients sequence is also periodic, we have a similar result for the shift in the DFS coefficients by an integer ℓ

, i.e.

$$\text{If } \tilde{x}[n] \leftrightarrow \tilde{X}[k]$$

$$\text{then } W_N^{n\ell}\tilde{x}[n] \leftrightarrow \tilde{X}[k-\ell]$$

Properties of the DFS (cont'd)

Duality:

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$

then $\tilde{X}[n] \leftrightarrow N\tilde{x}[-k]$ (Prove later)

Symmetry

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$

then

$$\mathrm{Re}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_e[k]\left(=\frac{1}{2}\left(\tilde{X}[k]+\tilde{X}^*[-k]\right)\right)$$

$$j\,\mathrm{Im}\{\tilde{x}[n]\} \leftrightarrow \tilde{X}_o[k]\left(=\frac{1}{2}\left(\tilde{X}[k]-\tilde{X}^*[-k]\right)\right)$$

Also

$$\tilde{x}_e[n]=\frac{1}{2}\left(\tilde{x}[n]+\tilde{x}^*[-n]\right) \leftrightarrow \mathrm{Re}\{\tilde{X}[k]\}$$

$$\tilde{x}_o[n]=\frac{1}{2}\left(\tilde{x}[n]-\tilde{x}^*[-n]\right) \leftrightarrow j\,\mathrm{Im}\{\tilde{X}[k]\}$$

Properties of the DFS (cont'd)

Symmetry (cont'd):

If $\tilde{x}[n]$ is real, $\tilde{X}[k] = \tilde{X}^*[-k]$

then $\begin{cases} \left|\tilde{X}[k]\right| = \left|\tilde{X}[-k]\right| \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$

$$\begin{cases} \mathrm{Re}\left\{\tilde{X}[k]\right\} = \mathrm{Re}\left\{\tilde{X}[-k]\right\} \\ \mathrm{Im}\left\{\tilde{X}[k]\right\} = -\mathrm{Im}\left\{\tilde{X}[-k]\right\} \end{cases}$$

Proof of Duality

Proof:

Since $\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$, then $\tilde{x}[-n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{kn} \Rightarrow N\tilde{x}[-n]=\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{kn}$

Compare this to the analysis eqn: $\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}$, then we see that

$$N\tilde{x}[-k]=\sum_{k=0}^{N-1}\tilde{X}[n]W_N^{kn}$$

Properties of DFS: Periodic Convolution (Circular Convolution)

Periodic Convolution:

If $\tilde{x}_1[n], \tilde{x}_2[n]$ are periodic sequences with period N

then $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \leftrightarrow \tilde{X}_1[k]\tilde{X}_2[k]$

$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n] \leftrightarrow \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$$

These convolution equations look very similar to the ones from DTFT, but there is a very subtle difference here. That is, the sum in the convolution here is only for a single period!

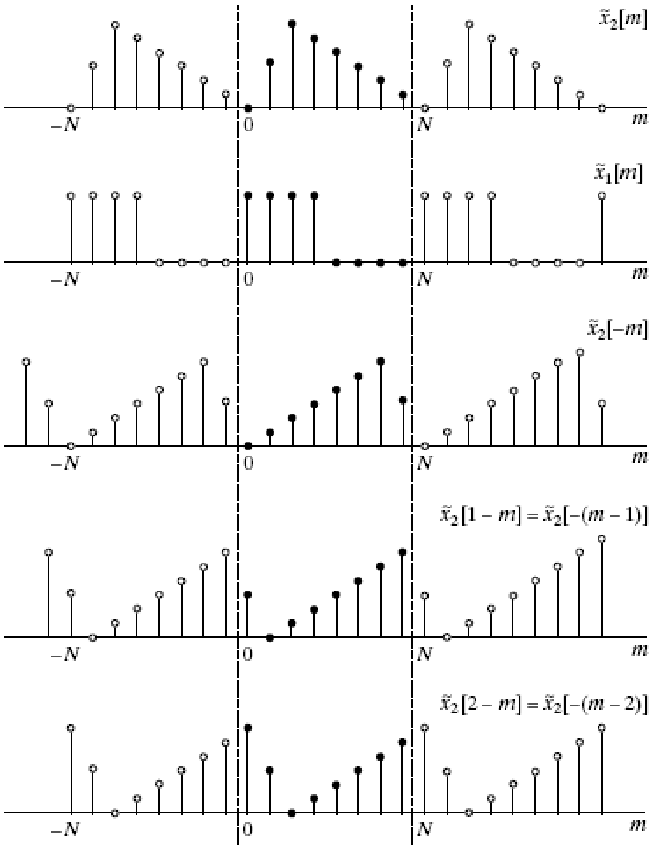


Figure 8.3 Procedure for forming the periodic convolution of two periodic sequences.

Properties of DFS

Discrete Fourier Series Properties

Periodic Sequence (Period N)	DFS Coefficients (Period N)
$\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
$\tilde{x}_1[n]$, $\tilde{x}_2[n]$	$\tilde{X}_1[k]$, $\tilde{X}_2[k]$ periodic with period N
$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
$\tilde{X}[n]$	$N\tilde{x}[-k]$ (Duality)
$\tilde{x}[n-m]$	$e^{-j\frac{2\pi}{N}km}\tilde{X}[k]$
$e^{j\frac{2\pi}{N}\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
$\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)

Properties of DFS

$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
$Re\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}\left(\tilde{X}[k] + \tilde{X}^*[-k]\right)$
$jIm\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}\left(\tilde{X}[k] - \tilde{X}^*[-k]\right)$
$\tilde{x}_e[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right)$	$Re\{\tilde{X}[k]\}$
$\tilde{x}_o[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right)$	$jIm\{\tilde{X}[k]\}$

The following properties apply only when $x[n]$ is real

Symmetry properties

$$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ Re\{\tilde{X}[k]\} = Re\{\tilde{X}[-k]\} \\ Im\{\tilde{X}[k]\} = -Im\{\tilde{X}[-k]\} \\ |\tilde{X}[k]| = |\tilde{X}[-k]| \\ \measuredangle\tilde{X}[k] = -\measuredangle\tilde{X}[-k] \end{cases}$$

$\tilde{x}_e[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}[-n]\right)$	$Re\{\tilde{X}[k]\}$
$\tilde{x}_o[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}[-n]\right)$	$jIm\{\tilde{X}[k]\}$

CTFT of Periodic Signals

Recall that the CTFT of a periodic continuous function $\tilde{x}(t)$ can be written as

$$\tilde{x}(t) = \sum_{m} x(t - mT_0) \quad \text{where} \quad x(t) = \begin{cases} \tilde{x}(t), & -\frac{T_0}{2} \le t \le \frac{T_0}{2} \\ 0, & else. \end{cases}$$

The CTFS of $\tilde{x}(t)$ can then be written as

$$\tilde{x}(t) = \sum_{n} c_n e^{jn\Omega_0 t} \quad \text{where} \quad c_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \tilde{x}(t) e^{-jn\Omega_0 t} dt \text{ and } \Omega_0 = 2\pi F_0 = \frac{2\pi}{T_0}.$$

Since the $x(t)$ is Fourier transformable, we may rewrite the CTFS coefficients c_n as

$$\begin{aligned} c_n &= F_0 \int_t x(t) e^{-jn\Omega_0 t} dt \\ &= F_0 X(j\Omega_0 n). \end{aligned}$$

Then we can rewrite the CTFS of $\tilde{x}(t)$ as

$$\tilde{x}(t) = F_0 \sum_{n} X(j\Omega_0 n) e^{j\Omega_0 nt},$$

which is equivalent to

$$\sum_{m} x(t - mT_0) = F_0 \sum_{n} X(j\Omega_0 n) e^{j\Omega_0 nt}.$$

CTFT of Periodic Signals (cont'd)

$$\sum_m x(t-mT_0) = F_0 \sum_n X(j\Omega_0 n) e^{j\Omega_0 nt}.$$

This is known as the Poisson's sum formula. Recalling that the CTFT of $e^{j\Omega_0 t}$ is $2\pi\delta(\Omega-\Omega_0)$. Then

$$\sum_m x(t-mT_0) \Leftrightarrow 2\pi F_0 \sum_n X(j\Omega_0 n)\delta(\Omega-\Omega_0 n) = \Omega_0 \sum_n X(j\Omega_0 n)\delta(\Omega-\Omega_0 n).$$

Therefore, the CTFT of a periodic continuous-time signal consists of delta functions (impulses) occurring at integer multiples of the fundamental frequency F_0, weighted by $\Omega_0 X(j\Omega_0 n)$, where $X(j\Omega_0 n)$ is the CTFT coefficient of the signal $x(t)$ evaluated at $\Omega_0 n$.

Fourier Transform of Periodic Signals

$$\sum_m x(t - mT_0) \Leftrightarrow \Omega_0 \sum_n X(j\Omega_0 n)\,\delta(\Omega - \Omega_0 n)$$

- Recall from Chapter 2 that the DTFT of a periodic sequence is represented by impulse train. This is because periodic signal has neither uniform convergence nor mean-square convergence since it is not absolutely summable nor square summable. As $n \rightarrow \pm\infty$, the sequence does not go to zero
- With the knowledge about the DFS, we will show that DFS is actually a sampled version of the DTFT (via examples below)
- Recall from above that the CTFT of a periodic signal requires the use of impulse train in the frequency domain with impulse values weighted by the CTFS coefficients for the signal
- Similarly, in discrete-time domain, the DTFT of a periodic sequence is represented by a impulse train in the frequency domain weighted by impulse values proportional to the DFS coefficients for the sequence, i.e.

$$\tilde{X}(e^{j\omega}) = \frac{2\pi}{N} \sum_k \tilde{X}[k]\,\delta\left(\omega - \frac{2\pi k}{N}\right)$$

- Note the $\tilde{X}(e^{j\omega})$ has periodicity of 2π because $\tilde{X}[k]$ is periodic with N and the impulses are spaced at integer multiples of $2\pi/N$

Example 1

Here, we want to show that the DFS coefficients are a sampled version of the DTFT.

Consider the periodic impulse train

$$\tilde{p}[n] = \sum_r \delta[n - rN].$$

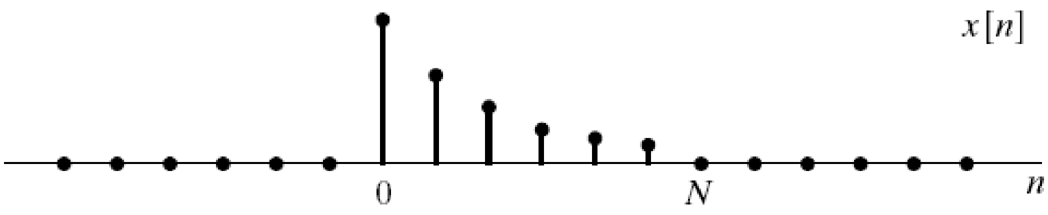
From the analysis equation of the DFS, we see that

$$\tilde{P}[k] = 1, \qquad \forall k.$$

Therefore, the Fourier transform of $\tilde{p}[n]$ is

$$\tilde{P}\left(e^{j\omega}\right) = \frac{2\pi}{N} \sum_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

Now consider the aperiodic sequence $x[n]$

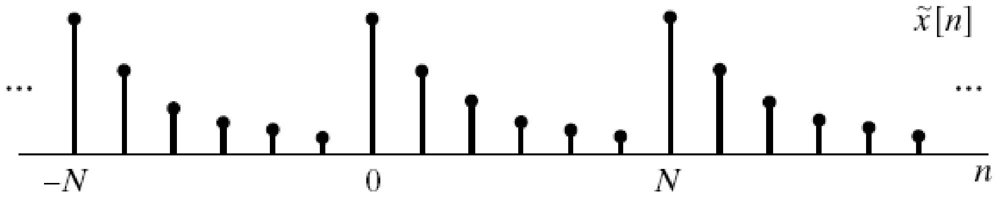


Example 1 (cont'd)

Now convolve $x[n]$ with $\tilde{p}[n]$ we get

$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_r \delta[n - rN]$$

$$= \sum_r x[n - rN].$$



Using the convolution property, the Fourier transform of $\tilde{x}[n]$ is

$$\tilde{X}\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)\tilde{P}\left(e^{j\omega}\right)$$

$$= X\left(e^{j\omega}\right)\frac{2\pi}{N}\sum_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$= \frac{2\pi}{N}\sum_k X\left(e^{j\frac{2\pi}{N}k}\right)\delta\left(\omega - \frac{2\pi k}{N}\right).$$

Example 1 (cont'd)

Therefore, comparing this equation to $\tilde{X}\left(e^{j\omega}\right)=\dfrac{2\pi}{N}\sum_k \tilde{X}[k]\delta\left(\omega-\dfrac{2\pi k}{N}\right)$,

we see that

$$\tilde{X}[k]=X\left(e^{j\frac{2\pi}{N}k}\right)=X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}}.$$

From this example, we see that the DFS coefficient $\tilde{X}[k]$ of the signal $\tilde{x}[n]$ is equal to the DTFT of the corresponding aperiodic signal $x[n]$, sampled at $\omega=\dfrac{2\pi k}{N}$.

Since the DTFT of $x[n]$ is periodic in ω with 2π, the DFS of $\tilde{x}[n]$ resulting sequence is periodic in k with period N

Example 2

Consider the periodic sequence $\tilde{x}[n]$

$$\tilde{x}[n] = \begin{cases} 1, & r10 \le n \le 4 + r10 \\ 0, & 5 + r10 \le n \le 9 + r10 \end{cases}$$

with $r \in \mathbb{Z}$. It is obvious that a single period of $\tilde{x}[n]$ is

$$x[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & else \end{cases} \quad , \quad \text{with } N = 10$$

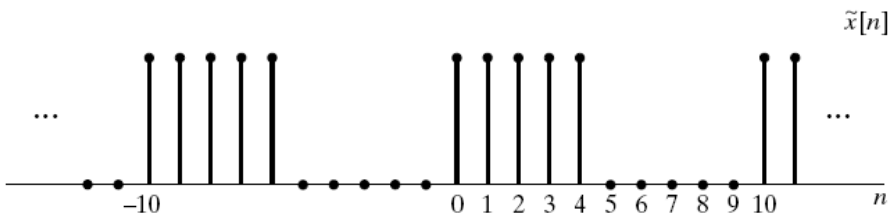


Figure 8.1 Periodic sequence with period $N = 10$ for which the Fourier series representation is to be computed.

Example 2 (cont'd)

The DTFT of the sequence is

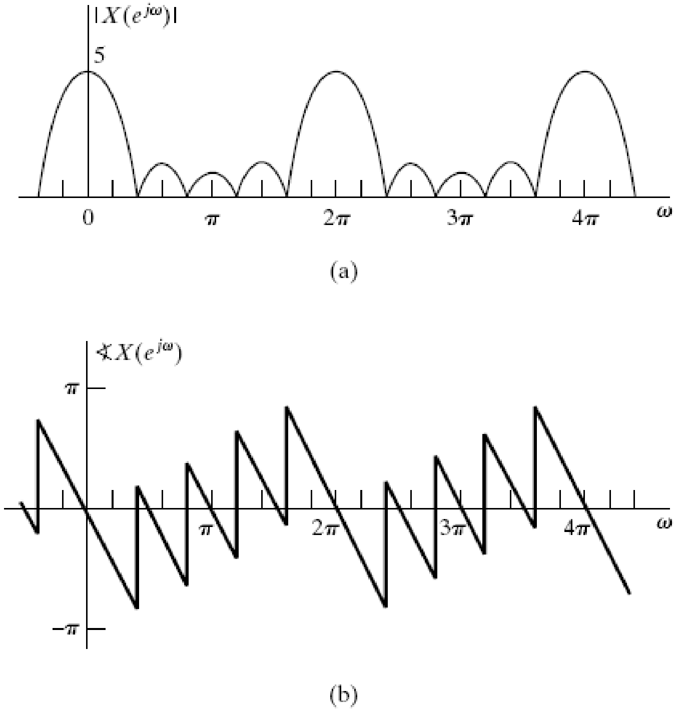


Figure 8.5 Magnitude and phase of the Fourier transform of one period of the sequence in Figure 8.1.

Example 2 (cont'd)

The Fourier transform of $x[n]$ is given by

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

Since $\tilde{X}[k] = X\left(e^{j\frac{2\pi}{N}k}\right) = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}}$, we substitute $\omega = 2\pi k/10$ (since $N = 10$, not 4) to obtain the DFS coefficients of $\tilde{x}[n]$, which is

$$\tilde{X}[k] = e^{-j\frac{2(2\pi k)}{10}}\frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$

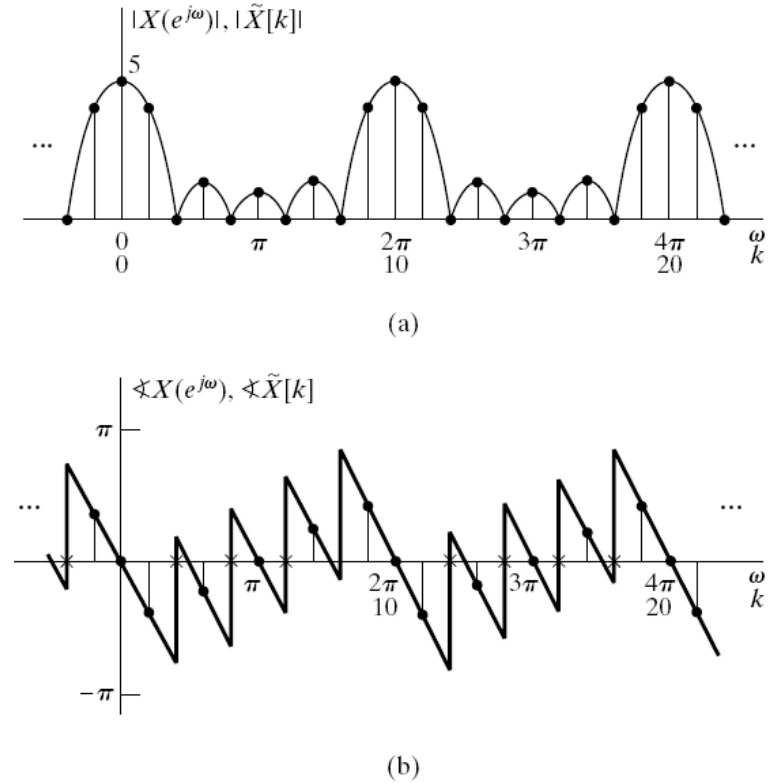


Figure 8.6 Overlay of Figures 8.2 and 8.5 illustrating the DFS coefficients of a periodic sequence as samples of the Fourier transform of one period.

Sampling the Fourier Transform

- As seen from the above, $\tilde{X}[k]$ is periodic in *k* with period *N*. Since the Fourier transform is equal to the *z*-transform evaluated on the unit circle, $\tilde{X}[k]$ can be obtained by sampling *X*(*z*) at *N* equally spaced points on the unit circle

$$\tilde{X}[k] = X(z)\big|_{z=e^{j\frac{2\pi k}{N}}} = X\left(e^{j\frac{2\pi k}{N}}\right)$$

- We have seen that $\tilde{X}[k]$ is the DFS coefficient of $\tilde{x}[n]$ and $\tilde{X}[k]$ is a sampled version of the DTFT of *x*[*n*]. So we formally derive the relationship between *x*[*n*] and $\tilde{x}[n]$ (as we have seen before, *x*[*n*] represents one of period of $\tilde{x}[n]$).
- We can actually draw parallels in the relationship between *x*[*n*] and $\tilde{x}[n]$ with that of $X(j\Omega)$ and $X(e^{j\omega})$

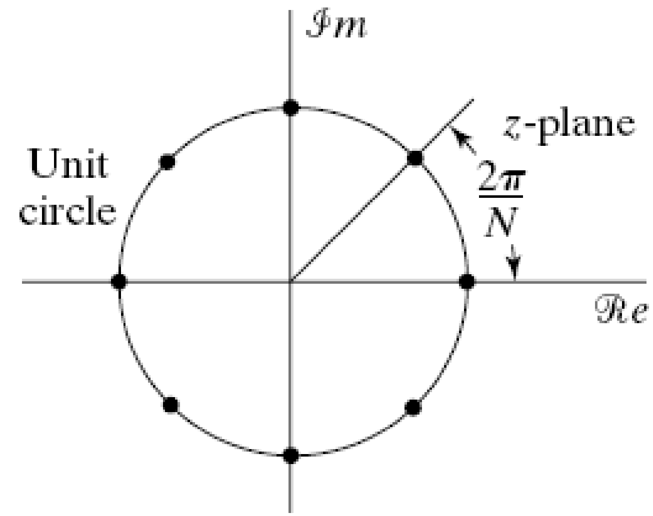


Figure 8.7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Comparison With Uniform Sampling in Time

Aperiodic sequence:

$$
\begin{array}{ccc}
x[n] & \rightarrow \text{DTFT} \rightarrow & X\left(e^{j\omega}\right) \\
\updownarrow ? & & \downarrow \text{sampling} \\
\tilde{x}[n] \leftarrow \text{IDFS} \leftarrow & \tilde{X}[k] = & X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{N}k}
\end{array}
$$

Compare to:

$$
\begin{array}{ccc}
x(t) & \rightarrow \text{CTFT} \rightarrow & X(j\Omega) \\
\downarrow \text{sampling} & & \updownarrow ? \\
x[n] & \rightarrow \text{DTFT} \rightarrow & X\left(e^{j\omega}\right)
\end{array}
$$

Relationship Between $x[n]$ and $\tilde{x}[n]$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} \qquad \text{(IDFS)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{N}k}\right)W_N^{-kn} \qquad \text{(Sampling)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)\Bigg|_{\omega=\frac{2\pi}{N}k} W_N^{-kn} \qquad \text{(FT)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\frac{2\pi}{N}km}\right)W_N^{-kn}$$

$$= \sum_{m} x[m]\frac{1}{N}\left[\sum_{k=0}^{N-1} W_N^{km}W_N^{-kn}\right]$$

$$= \sum_{m} x[m]\underbrace{\frac{1}{N}\left[\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right]}_{\sum_r \delta[n-m+rN]} \qquad \text{(recall that } \frac{1}{N}\sum_{k=0}^{N-1} W_N^{\ell k} = \sum_m \delta[\ell - mN]\text{)}$$

$$= x[n] * \sum_r \delta[n+rN]$$

$$= \sum_r x[n+rN]$$

Aliasing in Time

- In Figure 8.8, we show that if we sample $X(e^{j\omega})$ with high enough N, then $x[n]$ can be recovered from $\tilde{x}[n]$ by extracting one period of $\tilde{x}[n]$. But if we do not sample with enough N (Figure 8.9), then we will have aliasing in time. In that case, we will not be able to recover $x[n]$ from $\tilde{x}[n]$
- If $x[n]$ has finite length and we take a sufficient number of equally spaced samples of its Fourier Transform (a number greater than or equal to the length of $x[n]$), then $x[n]$ is recoverable from $\tilde{x}[n]$
- Two ways (equivalently to define the DFT):
  1) N samples of the DTFT of a finite duration sequence $x[n]$
  2) Or, make the period replica of $x[n] \rightarrow \tilde{x}[n]$
     Take the DFS of $\tilde{x}[n]$
     Pick up one segment of $\tilde{X}[k]$

$$\begin{array}{ccc} x[n] & \rightarrow DFT \rightarrow & X[k] \\ \downarrow \text{ periodic} & & \uparrow \text{ one segment} \\ \tilde{x}[n] & \rightarrow DFS \rightarrow & \tilde{X}[k] \end{array}$$

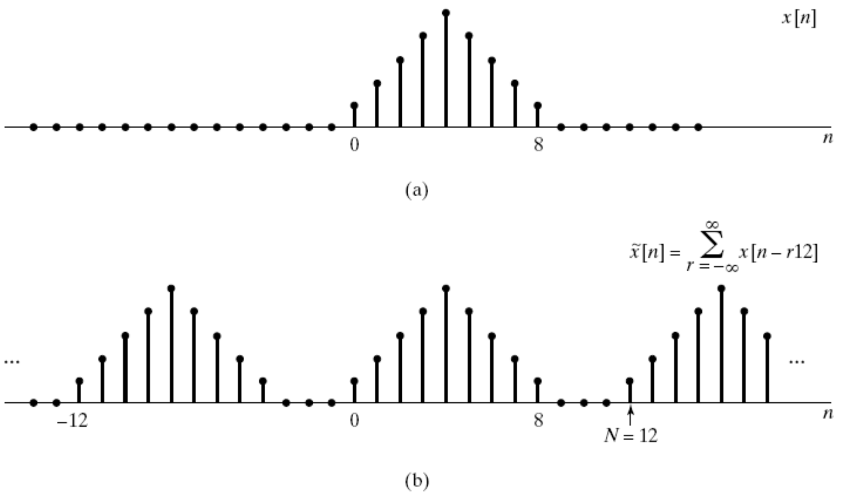
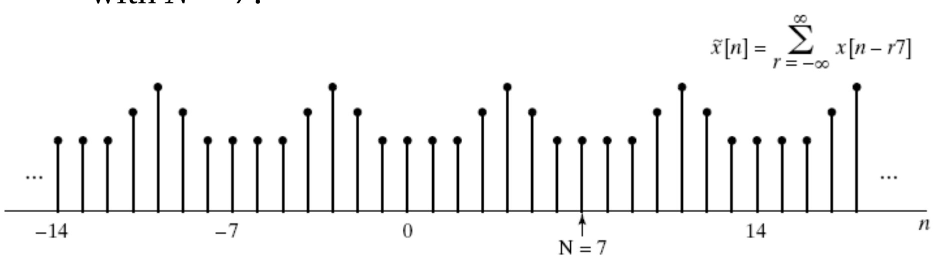


Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.
Figure 8.9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ in Figure 8.8(a) with $N = 7$.



Discrete Fourier Transform (DFT)

$x[n]$: length N, $0 \le n \le N-1$

Make the periodic replica:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN]$$

$$\equiv x[(n \text{ modulo } N)]$$

$$\equiv x[((n))_N]$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Keep one segment (finite duration)

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases} \quad \text{That is, } \tilde{X}[k] = X[((k))_N]$$

DFT (cont'd)

$$\textit{Analysis eqn}: \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

$$\textit{Synthesis eqn}: \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

Remarks:

- DFT formula is the same as DFS formula. Indeed, many properties of DFT are derived from those of DFS.

- Keep in mind that $X[k]$ is equal to samples of the $X(e^{j\omega})$, and if the synthesis equation is evaluated outside the interval $0 \le n \le N\text{-}1$, then the result will not be zero but a periodic extension of $x[n]$. This inherent periodicity is always present!

DFT Example

Consider the sequence: $x[n]=\begin{cases}1, & 0\le n\le 4\\ 0, & else\end{cases}$

Let $\tilde{x}[n]$ be the periodic extension of $x[n]$.

For the first case, let

$$\tilde{x}[n]=\begin{cases}1, & 5r\le n\le 4+5r\\ 0, & else\end{cases}$$

and for the second case, let

$$\tilde{x}[n]=\begin{cases}1, & 10r\le n\le 4+10r\\ 0, & else\end{cases}$$

In two cases, we can see that if we sample faster, then more information about the original DTFT can be *shown*, i.e. both DFTs (N=5 and N=10) contain same amount of information, but the extra information shown in the N=10 case is just hidden in the N=5 case. Remember, to get the 10-point DFT, we have simply zero-padded the original sequence, nothing more.

DFT Example (cont'd)

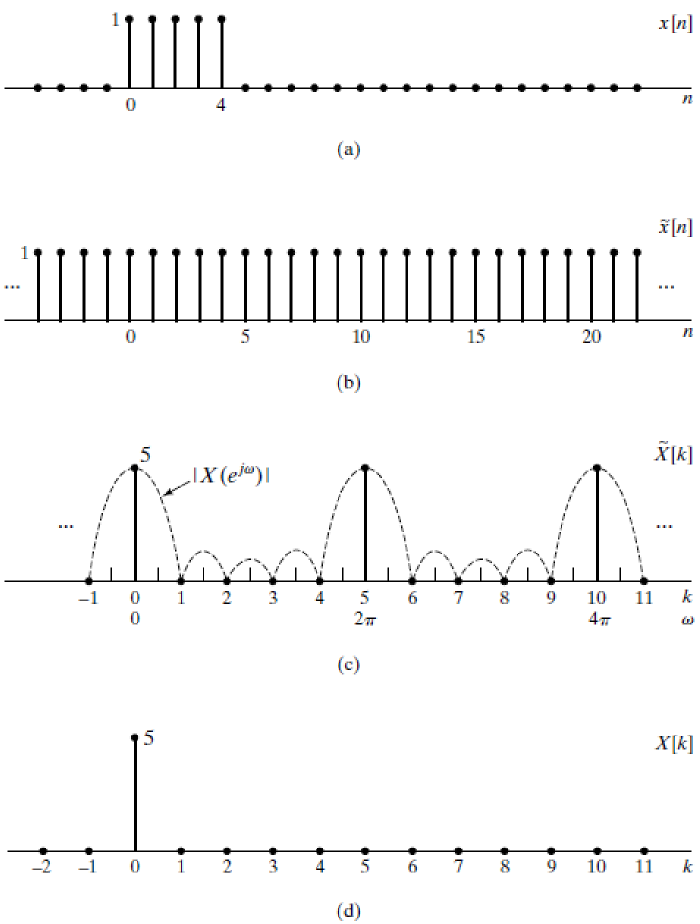


Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

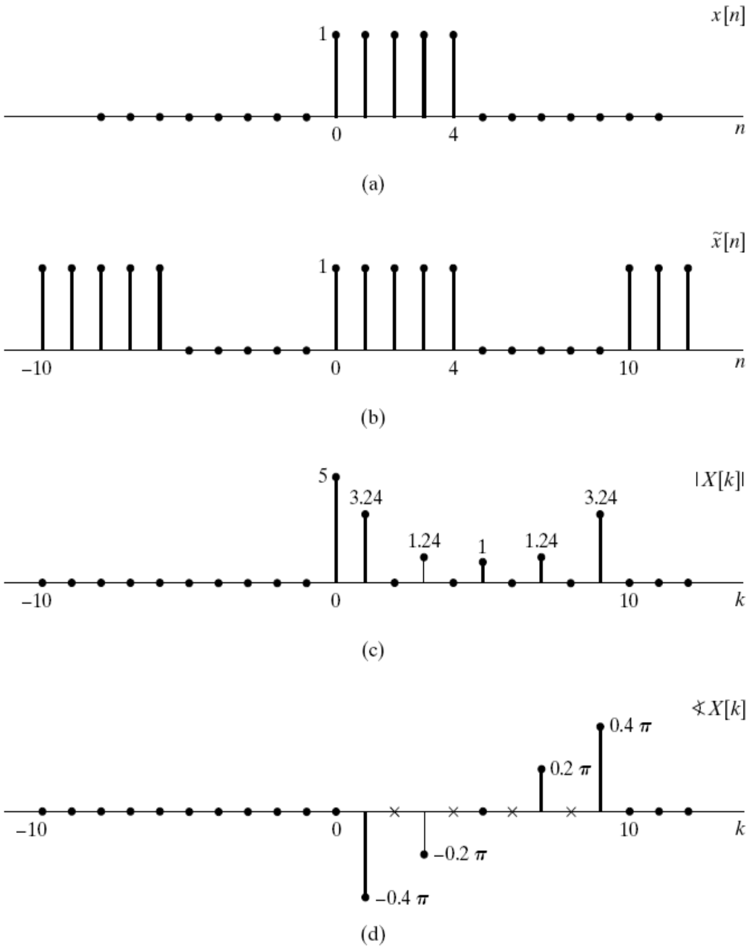


Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)

IDFT Matrix

- The IDFT and DFT is a linear transform
  - The computation can thus be represented using a matrix multiplication
- The (n,k)th element of an $N\times N$ matrix N-point IDFT matrix $\mathbf{W}_N$ is $[\mathbf{W}_N]_{nk} \triangleq \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn}$
  This is a symmetric (but complex-valued) matrix
- Example:

$$\mathbf{W}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{W}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

- DFT coeff. vector $\mathbf{x}_k = \mathbf{W}_N{}^H\mathbf{x}$, where $\mathbf{x}_k$ is a $N\times 1$ vector containing the DFT coefficients $X[k]$, and $\mathbf{x}$ is a $N\times 1$ vector containing the time domain sequence $x[n]$
- The DFT matrix can be generated by matlab using the function *dftmtx*. Note that this generates a *unnormalized* version of the DFT matrix, i.e. $\mathbf{W}_N$ is only an orthogonal matrix, NOT orthonormal, so $\mathbf{W}_N \mathbf{W}_N{}^H = N\mathbf{I}_N$ not $\mathbf{I}_N$
- Properties of the IDFT matrix
  - $\mathbf{W}_N = \mathbf{W}_N{}^T$, i.e. $\mathbf{W}_N$ is symmetric
  - $\mathbf{W}_N{}^H = \mathbf{W}_N{}^*$
  - $\mathbf{W}_N{}^{-1} = \mathbf{W}_N{}^*$

Properties of the DFT

Linearity:

$$\left.\begin{aligned}\tilde{x}_1[n] &\leftrightarrow X_1[k] \\ \tilde{x}_2[n] &\leftrightarrow X_2[k]\end{aligned}\right\}$$

then $ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k]$ (length= $\max[N_1, N_2]$

Circular Shift:

If $x[n] \leftrightarrow X[k]$

then $x\left[((n-m))_N\right] \leftrightarrow W_N^{km} X[k]$

also $W_N^{-\ell n} x[n] \leftrightarrow X\left[((k-\ell))_N\right]$

Proof of Circular Shift Property

Let

$$X_1[k] = e^{-j\frac{2\pi k}{N}m} X[k] = W_N^{km} X[k]$$

From the definition of the DFT, we know that

$$\tilde{x}_1[n] = x_1\left[((n))_N\right] \leftrightarrow \tilde{X}_1[k] = X_1\left[((k))_N\right]$$

Then, the DFS of $\tilde{x}_1[n]$ is

$$\tilde{X}_1[k] = e^{-j\frac{2\pi((k))_N}{N}m} X\left[((k))_N\right] = e^{-j\frac{2\pi k}{N}m} X\left[((k))_N\right] = e^{-j\frac{2\pi k}{N}m} \tilde{X}[k].$$

Then it follows that shifting property of the DFS that

$$\tilde{x}_1[n] = \tilde{x}[n-m].$$

Then from the definition of the DFT

$$\tilde{x}[n-m] = x\left[((n-m))_N\right].$$

Therefore,

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = x\left[((n-m))_N\right], & 0 \le n \le N-1 \\ 0, & else. \end{cases}$$

Remark: This is a circular shift, not linear shift. (Linear shift of a periodic sequence = circular shift of a finite sequence.)

Example: Circular Shift

We have $\tilde{x}[n] = x\left[((n))_6\right]$. We want to obtain $\tilde{x}_1[n] = \tilde{x}[n+2] = x\left[((n+2))_6\right]$. This is shown in Figure 8.12(c). Figure 8.12(d) shows a single period of $\tilde{x}_1[n]$.

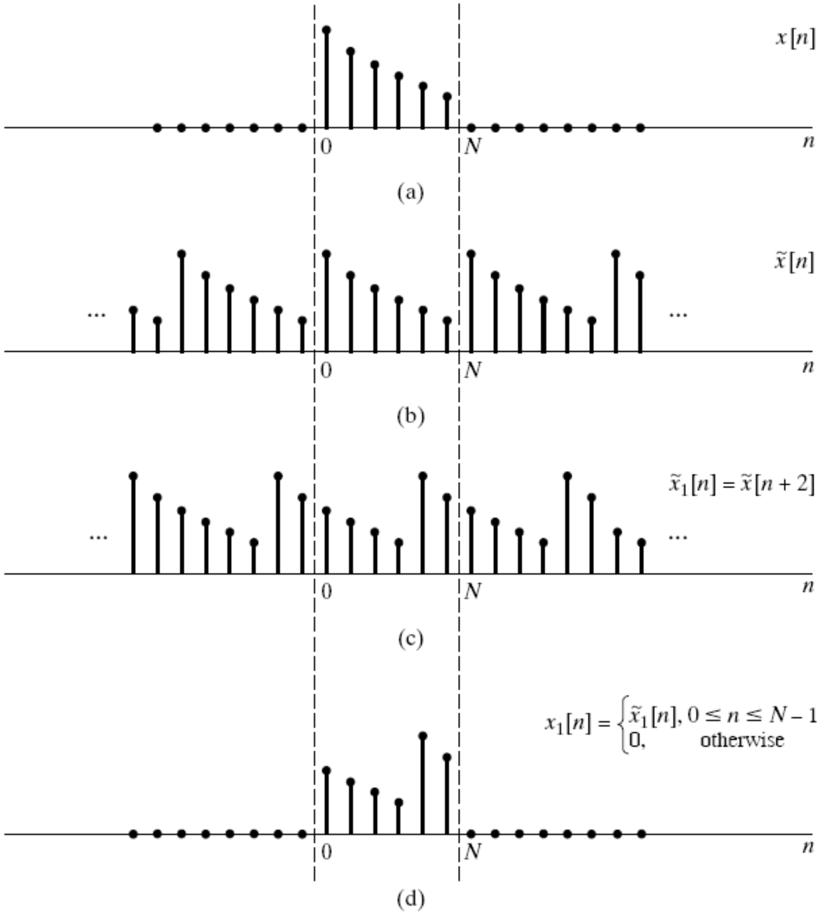


Figure 8.12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear phase factor.

Duality Property of the DFT

Duality:

If $x[n] \leftrightarrow X[k]$

then $X[n] \leftrightarrow Nx\left[((-k))_N\right], \quad 0 \le k \le N-1$

Example:

Consider the finite length sequence $x[n]$ shown in Figure 8.13(a). Figures 8.13(b) and 8.13(c) show the real and imaginary part of $X[k]$. To show the duality property, we relabel the k-axis in Figures 8.13(b) and (c) to be n-axis, shown in Figures 8.13(d) and (e). According to the duality property, Figure 8.13(f) shows the DFT of the complex-valued sequence of Figures 8.13(d) and (e).

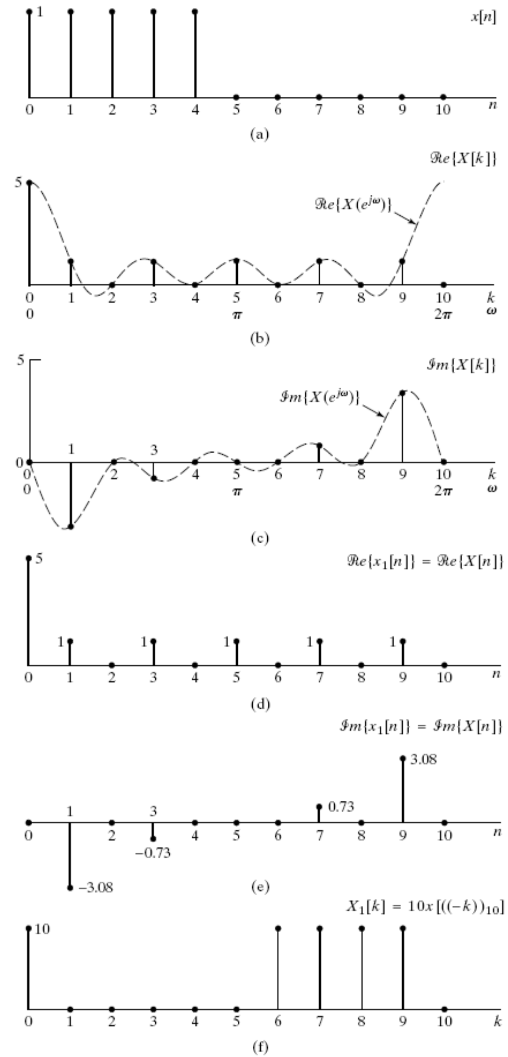


Figure 8.13 Illustration of duality. (a) Real finite-length sequence $x[n]$. (b) and (c) Real and imaginary parts of corresponding DFT $X[k]$. (d) and (e) The real and imaginary parts of the dual sequence $x_1[n] = X[n]$. (f) The DFT of $x_1[n]$.

Symmetry Properties of the DFT

Since $\tilde{x}[n] = x\left[((n))_N\right]$ and $\tilde{X}[k] = X\left[((k))_N\right]$, and $\tilde{x}^*[n] \leftrightarrow \tilde{X}^*[-k]$ and $\tilde{x}^*[-n] \leftrightarrow \tilde{X}^*[k]$, we have

$$x^*[n] \xleftrightarrow{DFT} X^*\left[((-k))_N\right], \quad 0 \le n \le N-1$$

$$x^*\left[((-n))_N\right] \xleftrightarrow{DFT} X^*[k], \quad 0 \le n \le N-1$$

Table 8.1: DFS properties

$Re\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}\left(\tilde{X}[k] + \tilde{X}^*[-k]\right)$
$jIm\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}\left(\tilde{X}[k] - \tilde{X}^*[-k]\right)$
$\tilde{x}_e[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right)$	$Re\{\tilde{X}[k]\}$
$\tilde{x}_o[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right)$	$jIm\{\tilde{X}[k]\}$

Symmetry Properties of the DFT

- It shows decomposition of periodic sequence into sum of a conjugate symmetric and a conjugate antisymmetric sequence.
- This suggests decomposition of finite-duration sequence $x[n]$ into the two finite-duration sequences of duration N corresponding to one period of the conjugate symmetric and conjugate antisymmetric components of $\tilde{x}[n]$. Denote these components of $x[n]$ as $x_{ep}[n]$ and $x_{op}[n]$
- With $\tilde{x}[n] = x\left[((n))_N\right]$, the conjugate symmetric part being $\tilde{x}_e[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right)$,

  and conjugate antisymmetric part being $\tilde{x}_o[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right)$, define

$$x_{ep}[n] = \tilde{x}_e[n], \quad 0 \le n \le N-1$$

$$x_{op}[n] = \tilde{x}_o[n], \quad 0 \le n \le N-1$$

  I.e. both $x_{ep}[n]$ and $x_{op}[n]$ are finite-duration sequences

Symmetry Properties of the DFT

Then $$x_{ep}[n]=\frac{1}{2}\left(x\left[((n))_N\right]+x^*\left[((-n))_N\right]\right),\quad 0\le n\le N-1$$

$$x_{op}[n]=\frac{1}{2}\left(x\left[((n))_N\right]-x^*\left[((-n))_N\right]\right),\quad 0\le n\le N-1$$

Since both sequences are finite, and since $((-n))_N=(N-n)$ and $((n))_N=n$ for $0\le n\le N-1$ then we can write these sequences as

$$x_{ep}[n]=\begin{cases}\frac{1}{2}\left\{x[n]+x^*[N-n]\right\}, & 1\le n\le N\text{-}1\\ \operatorname{Re}\left\{x[0]\right\}, & n=0\end{cases}$$

$$x_{op}[n]=\begin{cases}\frac{1}{2}\left\{x[n]-x^*[N-n]\right\}, & 1\le n\le N\text{-}1\\ j\operatorname{Im}\left\{x[0]\right\}, & n=0\end{cases}$$

This avoids the use of $((n))_N$. They are also not equal to $x_e[n]\triangleq\frac{1}{2}\left(x[n]+x^*[-n]\right)=x_e^*[n]$ nor $x_o[n]=\frac{1}{2}\left(x[n]-x^*[-n]\right)=-x_o^*[-n]$, resp.

Symmetry Properties of the DFT

We also have

$$\tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n] \;\Rightarrow\; x[n] = \tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n], \quad \text{for } 0 \le n \le N-1$$

$$= x_{ep}[n] + x_{op}[n]$$

$$\Rightarrow \begin{cases} \mathrm{Re}\{x[n]\} \leftrightarrow X_{ep}[k] = \frac{1}{2}\left\{X\left[((k))_N\right] + X^*\left[((-k))_N\right]\right\} \\ j\,\mathrm{Im}\{x[n]\} \leftrightarrow X_{op}[k] = \frac{1}{2}\left\{X\left[((k))_N\right] - X^*\left[((-k))_N\right]\right\} \end{cases}$$

$$\Rightarrow \quad x_{ep}[n] \leftrightarrow \mathrm{Re}\{X[k]\} \qquad x_{op}[n] \leftrightarrow j\,\mathrm{Im}\{X[k]\}$$

Symmetry Properties of the DFT

$$\text{If } x[n] \text{ real}, \quad X[k] = X^*\left[((-k))_N\right], \quad 0 \le k \le N-1$$

$$\Rightarrow \begin{cases} \left|X[k]\right| = \left|X\left[((-k))_N\right]\right| \\ \angle\{X[k]\} = -\angle X\left[((-k))_N\right] \end{cases}$$

$$\Rightarrow \begin{cases} \operatorname{Re}\{X[k]\} = \operatorname{Re}\left\{X\left[((-k))_N\right]\right\} \\ \operatorname{Im}\{X[k]\} = -\operatorname{Im}\left\{X\left[((-k))_N\right]\right\} \end{cases}$$

$$\begin{cases} \operatorname{Re}\{x[n]\} \leftrightarrow X_{ep}[k] = \frac{1}{2}\left\{X\left[((k))_N\right] + X^*\left[((-k))_N\right]\right\} \\ \operatorname{Im}\{x[n]\} \leftrightarrow X_{op}[k] = \frac{1}{2}\left\{X\left[((k))_N\right] - X^*\left[((-k))_N\right]\right\} \end{cases}$$

Circular Convolution of the DFT

From the DFS circular convolution: $\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]$, circular convolution for the DFT

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \quad 0 \le n \le N-1$$

$$= \sum_{m=0}^{N-1} x_1\left[((m))_N\right] x_2\left[((n-m))_N\right], \quad 0 \le n \le N-1.$$

Since $((m))_N = m$ for $0 \le m \le N-1$, then

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2\left[((n-m))_N\right], \quad 0 \le n \le N-1.$$

$$= x_1[n] \; \textcircled{N} \; x_2[n]$$

$$= \sum_{m=0}^{N-1} x_2[m] x_1\left[((n-m))_N\right], \quad 0 \le n \le N-1$$

$$= x_2[n] \; \textcircled{N} \; x_1[n]$$

where $\textcircled{N}$ denotes N-point circular convolution

Also, $x_1[n] \; \textcircled{N} \; x_2[n] \leftrightarrow X_1[k]X_2[k]$

Example 1: Circular Convolution

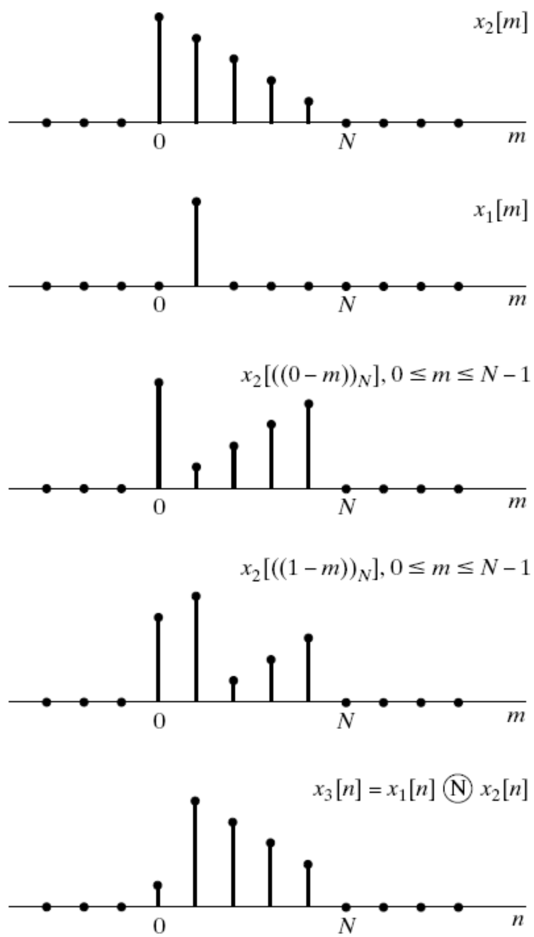


Figure 8.14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse, $x_1[n] = \delta[n-1]$.

Example 2: Circular Convolution (Aliasing)

N-point circular convolution of two constant sequences of length N. Let $L = 6$ for $x_1[n]$ and $x_2[n]$. Let $N = L$, then

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N, & k = 0, \\ 0, & else \end{cases}$$

$$X_3[k] = X_1[k] X_2[k] = \begin{cases} N^2, & k = 0, \\ 0, & else \end{cases}$$

$$\Leftrightarrow x_3[n] = N, \quad 0 \le n \le N-1$$

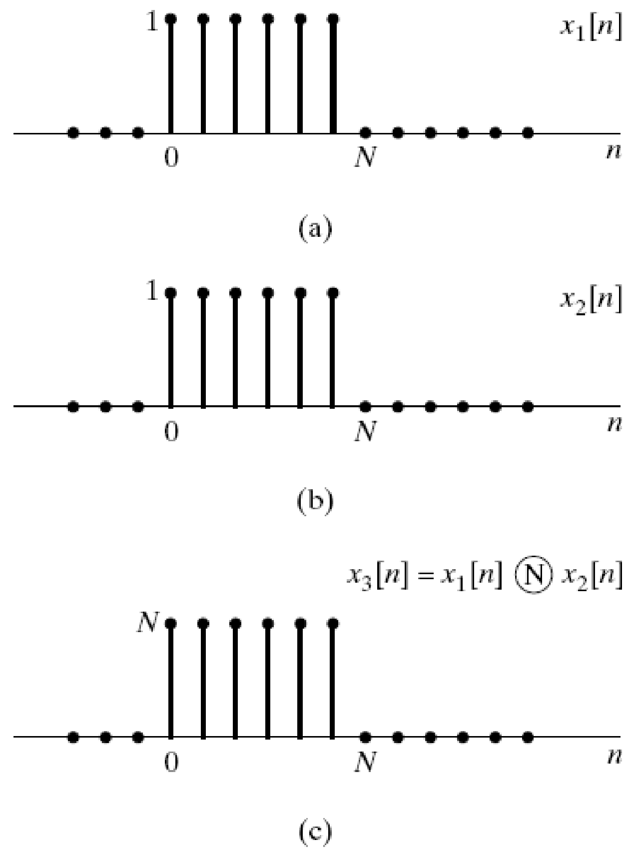


Figure 8.15 N-point circular convolution of two constant sequences of length N.

Note the result we get in Figure 8.15 is different from that of linear convolution

Example 2: Circular Convolution (Alias-free)

If we now perform a $N = 2L$-point circular convolution with $x_1[n]$ and $x_2[n]$ by appending L zeros to both sequences, then

$$X_1[k] = X_2[k] = \frac{1 - W_N^{Lk}}{1 - W_N^{k}}.$$

So the result becomes

$$X_3[k] = X_1[k]X_2[k] = \left(\frac{1 - W_N^{LK}}{1 - W_N^{k}}\right)^2,$$

where $N = L$.

Now the result in Figure 8.16 matches the result we would have obtained if we were to perform linear convolution between $x_1[n]$ and $x_2[n]$.
Therefore, unlike the relationship between linear convolution and multiplication for the DTFT, linear convolution and DFT multiplication may result in different sequences depending on the number of points of DFT we perform. This is due to the inherent periodic nature of the 2 sequences $x_1[n]$ and $x_2[n]$.

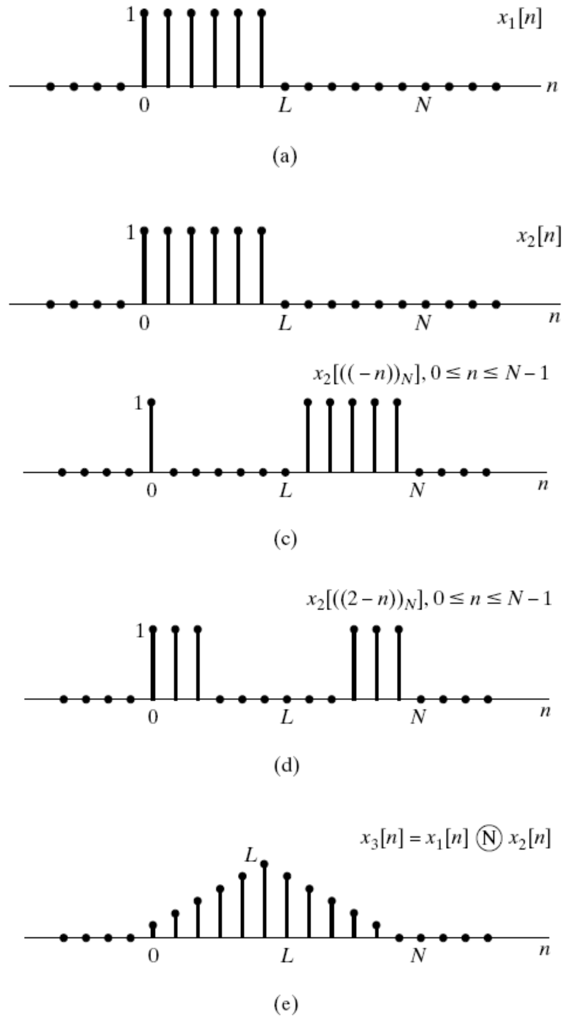


Figure 8.16 $2L$-point circular convolution of two constant sequences of length L.

Linear Convolution Using DFT

- Why use the DFT? There are fast DFT algorithms (FFT), so it might be more computational efficient to do all your processing in the transformed domain, followed by inverse operation to transform the result back into time-domain. However, we do not want the result of circular convolution, but rather, that of linear convolution. As seen in the preceding example, this can be accomplished by choosing the *appropriate* value for N.
- How do we do it?
  1. Compute the N-point DFT of $x_1[n]$ and $x_2[n]$, separately
  2. Compute the product $X_3[k] = X_1[k]X_2[k]$
  3. Compute the N-point IDFT of $X_3[k] \rightarrow x_3[n]$
- Problems
  - Aliasing
  - Very long sequence

Aliasing

- Let $x_1[n]$ be an length L sequence
- Let $x_2[n]$ be an length P sequence
- In order to avoid aliasing, $N \geq L+P-1$

$$x_{3p}[n] = x_1[n] \; \textcircled{N} \; x_2[n]$$

$$= \begin{cases} \sum_r x_3[n-rN], & 0 \le n \le N-1, \\ 0, & else \end{cases}$$

where $x_3[n]$ is the result of the linear convolution between $x_1[n]$ and $x_2[n]$. $x_{3p}[n]$ is of this form because of the inherent periodicity of $x_1[n]$ and $x_2[n]$. So when the circular convolution is performed, the tail of $x_1[n]/x_2[n]$ will wrap around to the head of $x_1[n]/x_2[n]$. Therefore, aliasing in time will occur if $N < L+P$.

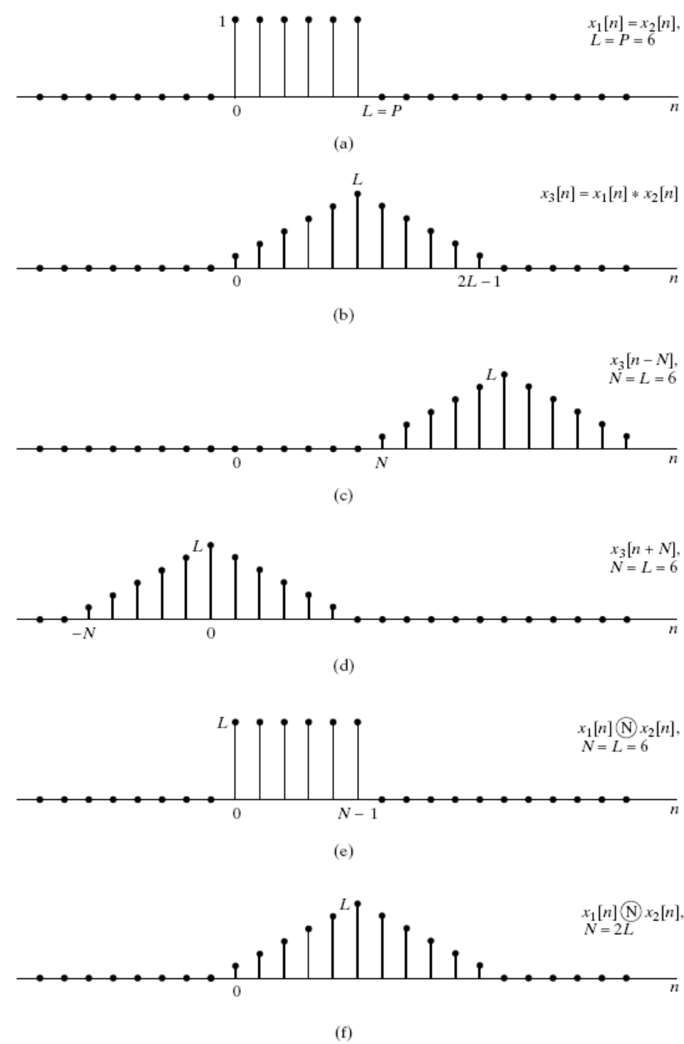


Figure 8.18 Illustration that circular convolution is equivalent to linear convolution followed by aliasing. (a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved. (b) The linear convolution of $x_1[n]$ and $x_2[n]$. (c) $x_3[n-N]$ for $N=6$. (d) $x_3[n+N]$ for $N=6$. (e) $x_1[n]$ ⑥ $x_2[n]$, which is equal to the sum of (b), (c), and (d) in the interval $0 \le n \le 5$. (f) $x_1[n]$ ⑫ $x_2[n]$.

Aliasing (Partial Distortion)

- If $N=L=P$, then all the samples of $x_{3p}[n]$ are corrupted by aliasing. However, if $P < L=N$, then only some of the samples in $x_{3p}[n]$ are corrupted, while the rest of it will equal to $x_3[n]$. Specifically, the first P-1 points of the result are incorrect, while the remaining points are identical to those that would be obtained from linear convolution (see Figure 8.21).

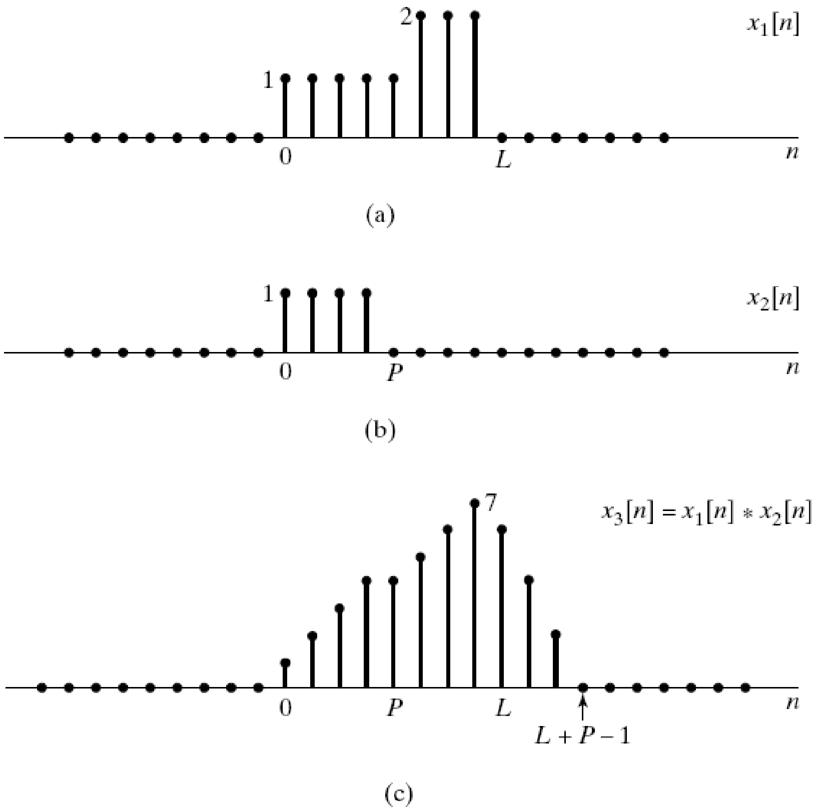


Figure 8.19 An example of linear convolution of two finite-length sequences.

Aliasing (cont'd)

- $x_1[n]$ pad with zeros → length N
- $x_2[n]$ pad with zeros → length N
- *Interpretation:* (Why call it aliasing?)
  - $X_3[k]$ has a (time domain) bandwidth of size $L+P$-1. That is, the nonzero values of $x_3[n]$ can be at most $L+P$-1. Therefore, $X_3[k]$ should have at least $L+P$-1 samples. If the sampling rate is insufficient, aliasing occurs on $x_3[n]$.

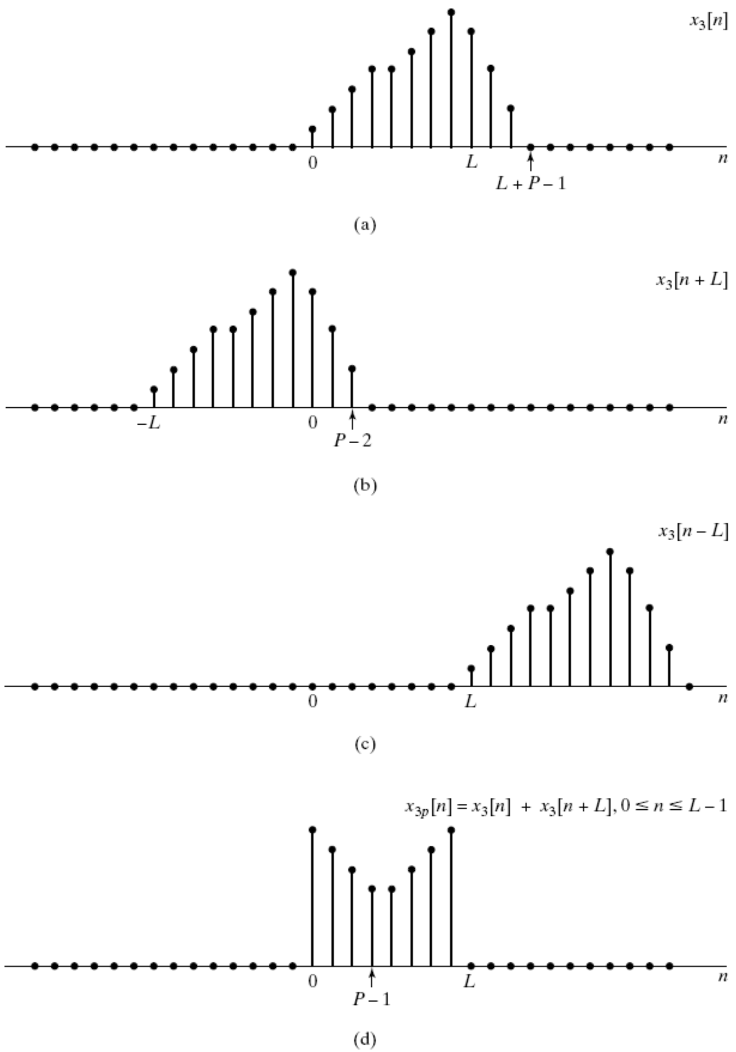


Figure 8.20 Interpretation of circular convolution as linear convolution followed by aliasing for the circular convolution of the two sequences $x_1[n]$ and $x_2[n]$ in Figure 8.19.

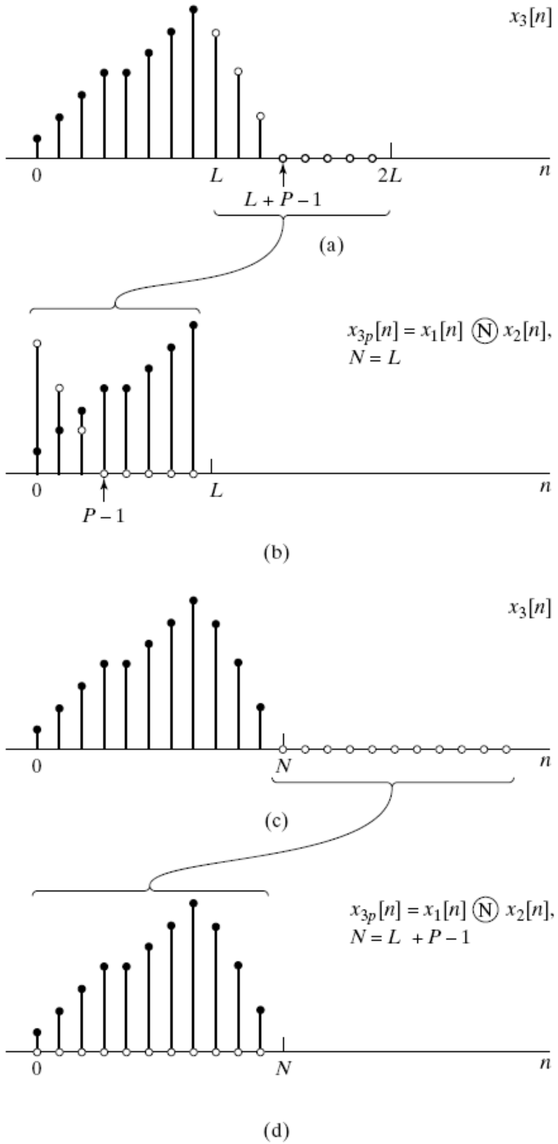


Figure 8.21 Illustration of how the result of a circular convolution "wraps around." (a) and (b) $N = L$, so the aliased "tail" overlaps the first $(P - 1)$ points. (c) and (d) $N = (L + P - 1)$, so no overlap occurs.

FIR Filtering

- How do we obtain output of filter if the input sequence is “infinitely” long (or unknown length)
  - E.g. speech recognition system
  - Solution (block convolution)
    - Partition input sequence into blocks
    - Perform convolution for each block of data
    - Somehow combine the results from each block processed
  - Efficient block convolution can be carried out in frequency domain
- Two methods
  - Overlap and add (overlap-add)
  - Overlap and save (overlap-save)

Overlap-Add

Partition the long sequence into non-overlapping sections of shorter length. For example, the filter impulse response $h[n]$ has finite length P and the input data $x[n]$ is nearly "infinite".

$$\text{Let } x[n] = \sum_{r=0}^{\infty} x_r[n - rL]$$

$$\text{where } x_r[n] = \begin{cases} x[n + rL], & 0 \le n \le L\text{-}1 \\ 0, & \text{otherwise} \end{cases}$$

The system (filter) output is a linear convolution. Since linear convolution is a time-invariant operation, then

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_k x[k] h[n-k] \\ &= \sum_k \sum_{r=0}^{\infty} x_r[k - rL] h[n-k] \\ &= \sum_{r=0}^{\infty} \underbrace{\sum_k x_r[k - rL] h[n-k]}_{x_r[n-rL]*h[n]=y_r[n-rL]} \\ &= \sum_{r=0}^{\infty} y_r[n - rL] \quad \text{where } y_r[n] = x_r[n] * h[n] \end{aligned}$$

Remark: The convolution length is $L+P$-1. That is, the $L+P$-1 point DFT is used. $y_r[n]$ has $L+P$-1 data points, among them, (P-1) points should be added to the next section. Hence, the input sequences are each padded with extra zeros at the end to make them both to have length $L+P$-1.

Overlap-Add (cont'd)

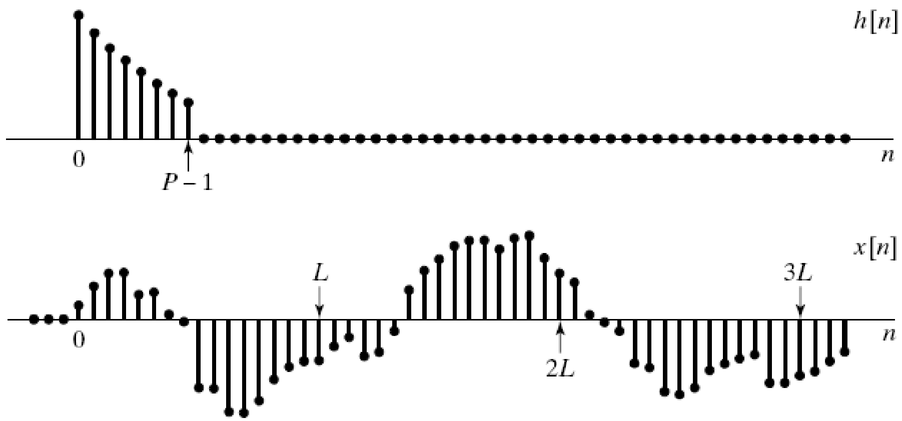


Figure 8.22 Finite-length impulse response $h[n]$ and indefinite-length signal $x[n]$ to be filtered.

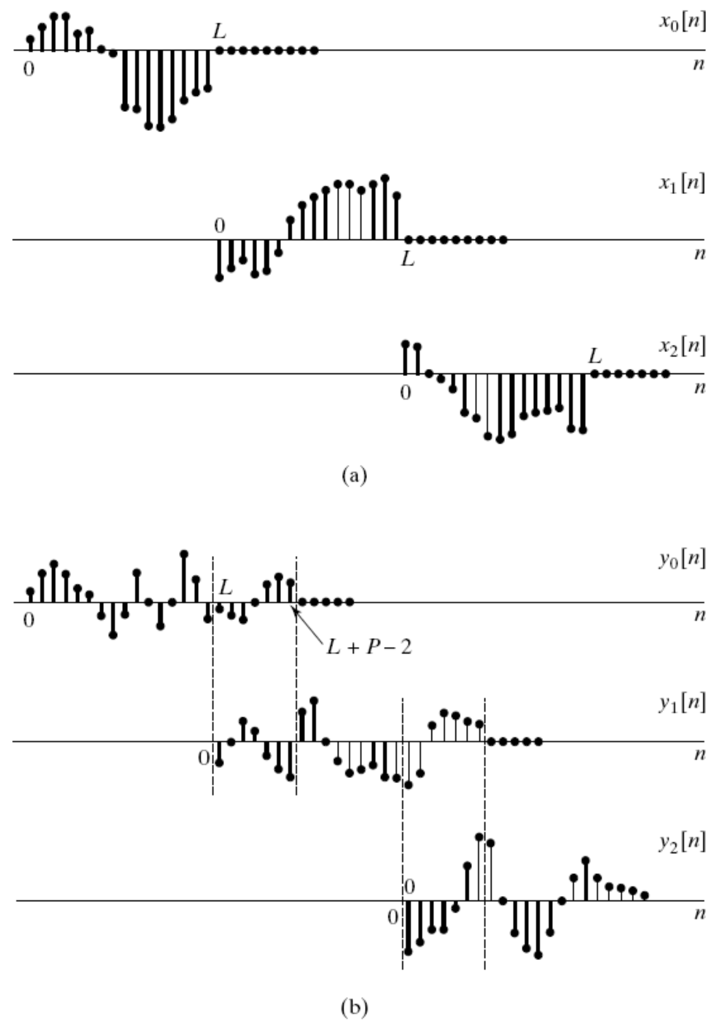


Figure 8.23 (a) Decomposition of $x[n]$ in Figure 8.22 into nonoverlapping sections of length L. (b) Result of convolving each section with $h[n]$.

Key idea: the input data are partitioned into *nonoverlapping* sections → the section outputs are overlapped and added together

Overlap-Save

- O&S implements an L-point (O&A implements an $L+P$-1 circular convolution) *circular convolution* for a length L and length P sequence, where $P < L$. Let $N = L$, then (as seen earlier) only the first P-1 points of the result are incorrect. Taking this into consideration, a new strategy is devised for filtering a long length signal (of length L).
- Steps
  1. Partition the long sequence into overlapping sections
  2. After computing DFT and IDFT, throw away some (incorrect) outputs
  3. For each section (length L, which is also the DFT size), we want to retain the correct data of length L-(P-1) points

Overlap-Save (cont'd)

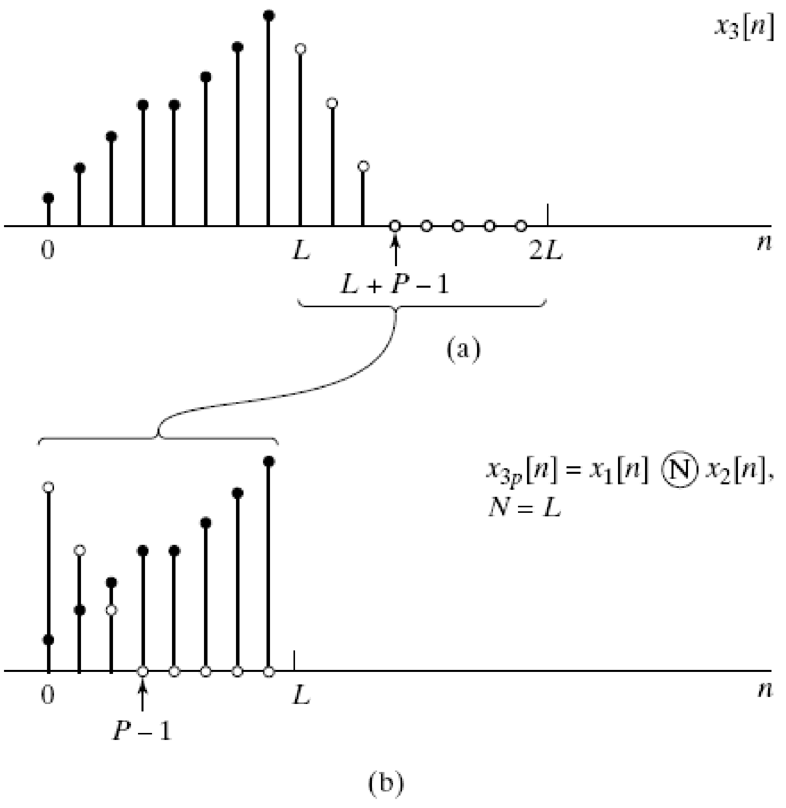
Let $h[n]$ be of length P.

Let $x_r[n]$ be of length L $(L > P)$

Then, $y_r[n]$ contains $P-1$ incorrect points at the beginning.

Therefore, we divide $x[n]$ into sections of length L but each section overlaps the preceding section by $P-1$ points, that is, we define the sequence of each section to be

$$x_r[n] = x[n + r(L - P + 1) - (P - 1)], \quad 0 \le n \le L\text{-}1$$



Overlap-Save (cont'd)

Then, circularly convolve each section, $x_r[n]$, with $h[n]$. Since the first P-1 samples are incorrect, after the (circular) convolution has taken place, simply discard these samples. The remaining samples from successive sections are put adjacent to each other to form the final output $y[n]$, i.e.

$$y[n] = \sum_{r=0}^{\infty} y_r\left[n - r\left(L - P + 1\right) + P - 1\right]$$

where

$$y_r[n] = \begin{cases} y_{rp}[n], & P-1 \le n \le L-1, \\ 0, & else \end{cases}$$

where $y_{rp}[n]$ is the result of the circular convolution for the r^{th} section.

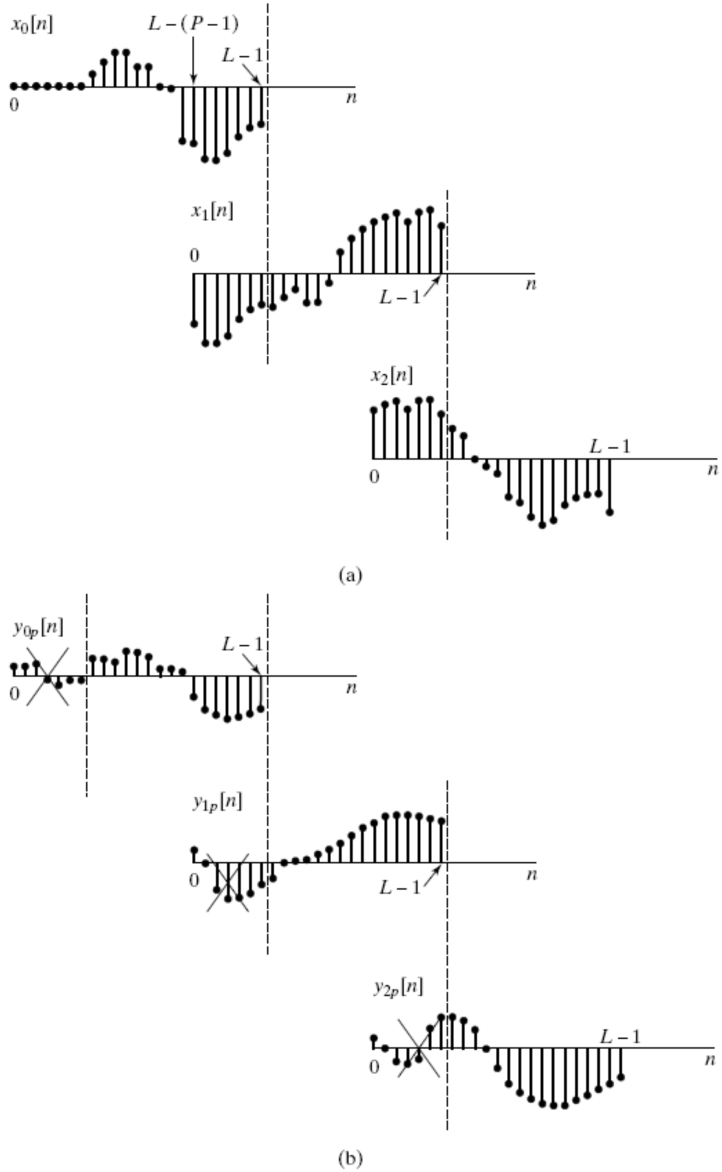


Figure 8.24 (a) Decomposition of $x[n]$ in Figure 8.22 into overlapping sections of length L. (b) Result of convolving each section with $h[n]$. The portions of each filtered section to be discarded in forming the linear convolution are indicated.

Overlap-Save: Matrix Representation

If you recall, assuming that $P = 3$ and $L = 5$, the linear convolution between $h[n]$ and $x[n]$ can be written in matrix form as

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} h[0] & & & & \\ h[1] & h[0] & & & \\ h[2] & h[1] & h[0] & & \\ & h[2] & h[1] & h[0] & \\ & & h[2] & h[1] & h[0] \\ & & & h[2] & h[1] \\ & & & & h[2] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$

$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

Since convolution is a commutative operation, we can also write

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} x[0] & & \\ x[1] & x[0] & \\ x[2] & x[1] & x[0] \\ x[3] & x[2] & x[1] \\ x[4] & x[3] & x[2] \\ & x[4] & x[3] \\ & & x[4] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix}$$

$$\mathbf{y} = \mathbf{X}\mathbf{h}$$

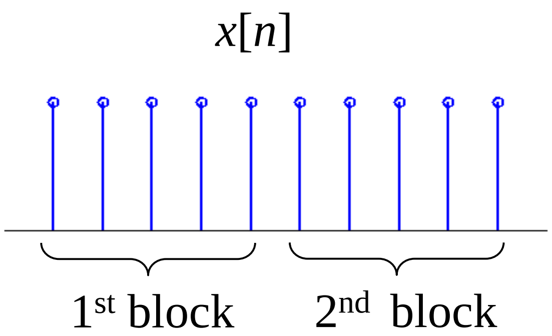
Overlap-Save: Matrix Representation (cont'd)

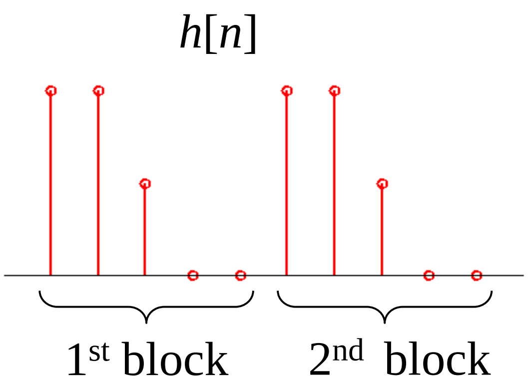
$$y[n] = \sum_{m=0}^{N-1} x[m] h\left[((n-m))_N\right], \quad 0 \le n \le N-1.$$

$$= x[n] \; \textcircled{N} \; h[n]$$

$$= \sum_{m=0}^{N-1} h[m] x\left[((n-m))_N\right], \quad 0 \le n \le N-1$$

$$= h[n] \; \textcircled{N} \; x[n]$$



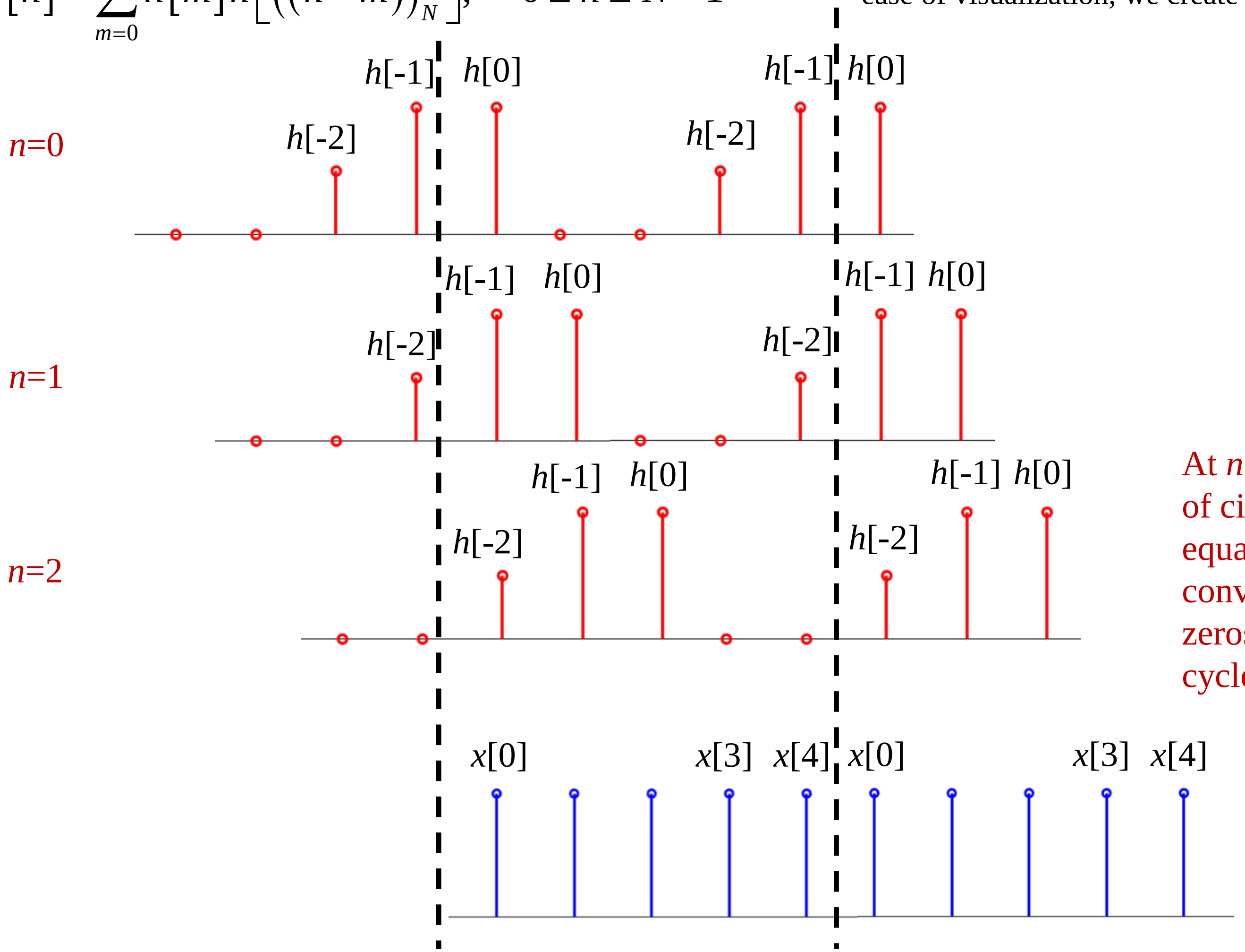


For $N = L$, the N-point circular convolution will be equal to the linear convolution except for the initial P-1 points in the result. The figures above show the (periodic extension with 2 periods only) impulse responses of $x[n]$ (blue) and $h[n]$ (red). Therefore, the circular convolution can be written in matrix form as

Overlap-Save: Matrix Representation (cont'd)

$$y[n] = \sum_{m=0}^{N-1} x[m] h\left[((n-m))_N\right], \quad 0 \le n \le N-1$$

$h[n]$ is regarded as a finite-duration sequence, but for ease of visualization, we create another cycle



At n = 2 (3 and 4), output of circular convolution equals to that of linear convolution due to the 2 zeros inserted in each cycle

Overlap-Save: Matrix Representation (cont'd)

add

$$\begin{bmatrix} \tilde{y}[0] \\ \tilde{y}[1] \\ \tilde{y}[2] \\ \tilde{y}[3] \\ \tilde{y}[4] \end{bmatrix} = \begin{bmatrix} h[2] & h[1] & h[0] & & & & \\ & h[2] & h[1] & h[0] & & & \\ & & h[2] & h[1] & h[0] & & \\ & & & h[2] & h[1] & h[0] & \\ & & & & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[3] \\ x[4] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$

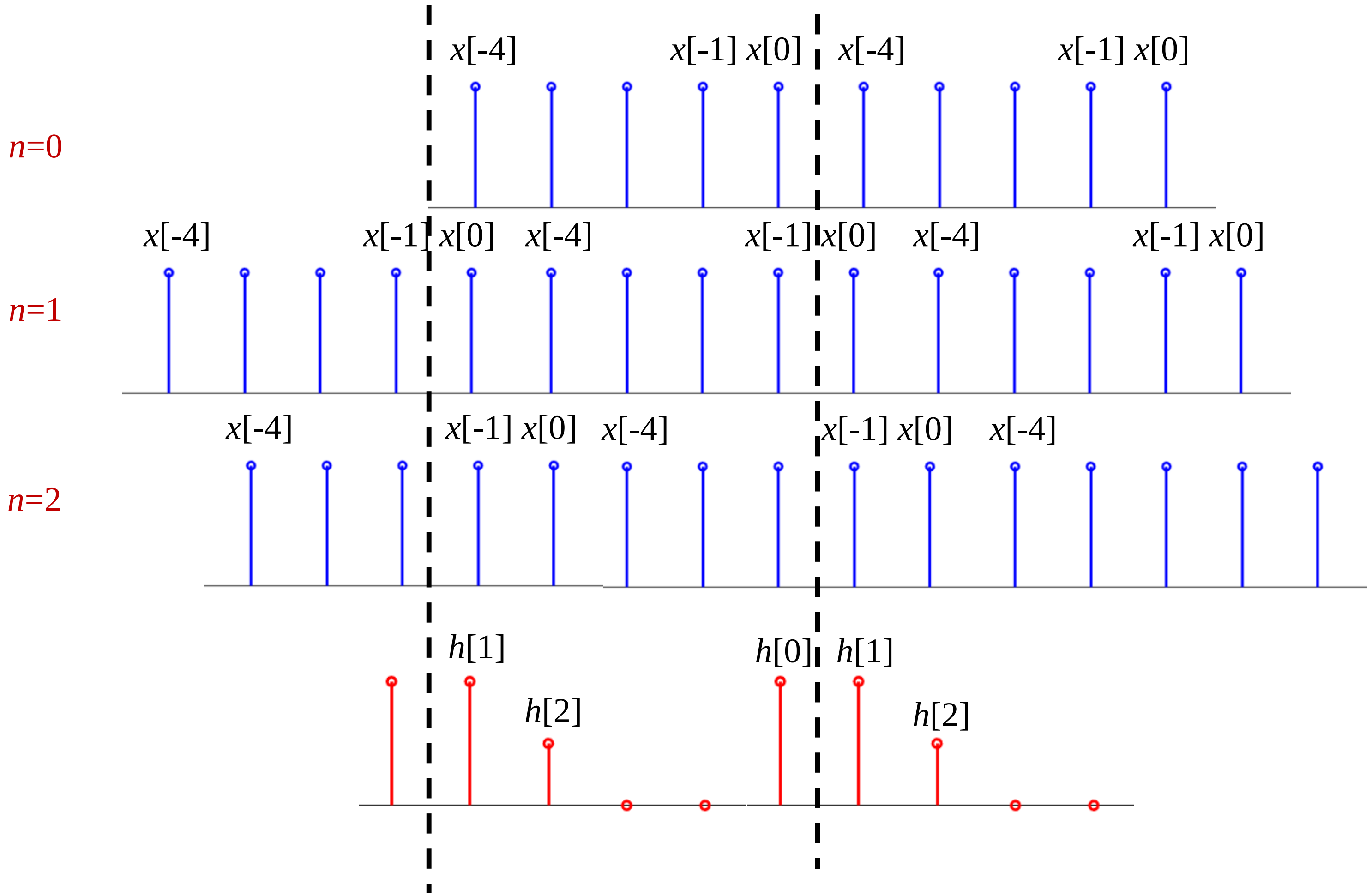
$$\Leftrightarrow \begin{bmatrix} \tilde{y}[0] \\ \tilde{y}[1] \\ \tilde{y}[2] \\ \tilde{y}[3] \\ \tilde{y}[4] \end{bmatrix} = \begin{bmatrix} h[0] & & & h[2] & h[1] \\ h[1] & h[0] & & & h[2] \\ h[2] & h[1] & h[0] & & \\ & h[2] & h[1] & h[0] & \\ & & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix} = \tilde{\mathbf{y}}$$

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\mathbf{x}$$

Overlap-Save: Matrix Representation (cont'd)

$$y[n]=\sum_{m=0}^{N-1} h[m]x\left[\left((n-m)\right)_N\right], \quad 0 \le n \le N-1$$

$x[n]$ is regarded as a finite-duration sequence, but for ease of visualization, we create another cycle



Overlap-Save: Matrix Representation (cont'd)

Due to the commutative property of the convolution, I can also write this as

shift

$$\begin{bmatrix} \tilde{y}[0] \\ \tilde{y}[1] \\ \tilde{y}[2] \\ \tilde{y}[3] \\ \tilde{y}[4] \end{bmatrix} = \begin{bmatrix} x[4] & x[3] & x[2] & x[1] & x[0] \\ x[0] & x[4] & x[3] & x[2] & x[1] \\ x[1] & x[0] & x[4] & x[3] & x[2] \\ x[2] & x[1] & x[0] & x[4] & x[3] \\ x[3] & x[2] & x[1] & x[0] & x[4] \end{bmatrix} \begin{bmatrix} h[1] \\ h[2] \\ 0 \\ 0 \\ h[0] \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \tilde{y}[0] \\ \tilde{y}[1] \\ \tilde{y}[2] \\ \tilde{y}[3] \\ \tilde{y}[4] \end{bmatrix} = \begin{bmatrix} x[0] & x[4] & x[3] & x[2] & x[1] \\ x[1] & x[0] & x[4] & x[3] & x[2] \\ x[2] & x[1] & x[0] & x[4] & x[3] \\ x[3] & x[2] & x[1] & x[0] & x[4] \\ x[4] & x[3] & x[2] & x[1] & x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ 0 \\ 0 \end{bmatrix} = \tilde{\mathbf{y}}$$

$$\tilde{\mathbf{y}} = \tilde{\mathbf{X}}\tilde{\mathbf{h}}$$

Overlap-Save: Matrix Representation (cont'd)

Note that $\tilde{y}[0]$ and $\tilde{y}[1]$ are corrupted by samples from the previous block. These are the first $P-1$ samples of the output. Only $y[2]=\tilde{y}[2]$, $y[3]=\tilde{y}[3]$ and $y[4]=\tilde{y}[4]$ are valid. Also, $\tilde{y}[5]$ and $\tilde{y}[6]$ (not shown) will be corrupted by the next block, so they will be discarded as well.

For any circulant matrix $\tilde{\mathbf{H}}$, it can be diagonalized by IFFT matrix $\mathbf{W}_N$, i.e.

$\tilde{\mathbf{H}} = \mathbf{W}_N \boldsymbol{\Lambda}_H \mathbf{W}_N^H$, where the (k,n) element of $\mathbf{W}_N$ is $[\mathbf{W}_N]_{kn} = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn}$, for $k,n = 0,1,\ldots N$ -1. That is, the diagonal elements of $\boldsymbol{\Lambda}_H$ contains the DFT coefficients of the first column vector of $\tilde{\mathbf{H}}$.

Note: This can always be done regardless of the values of $\tilde{\mathbf{H}}$ as long as $\tilde{\mathbf{H}}$ is circulant, i.e. structure requirement is the only requirement.