Introduction to Multirate System for Signal Analysis

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Outline

- Good old Fourier: Strengths and Weaknesses
- Multirate System Basics and PR
- STFT and Wavelets: Transform Interpretation
- Implementation of STFT via Filter Banks
- Construction and Implementation of Wavelets via Filter Banks

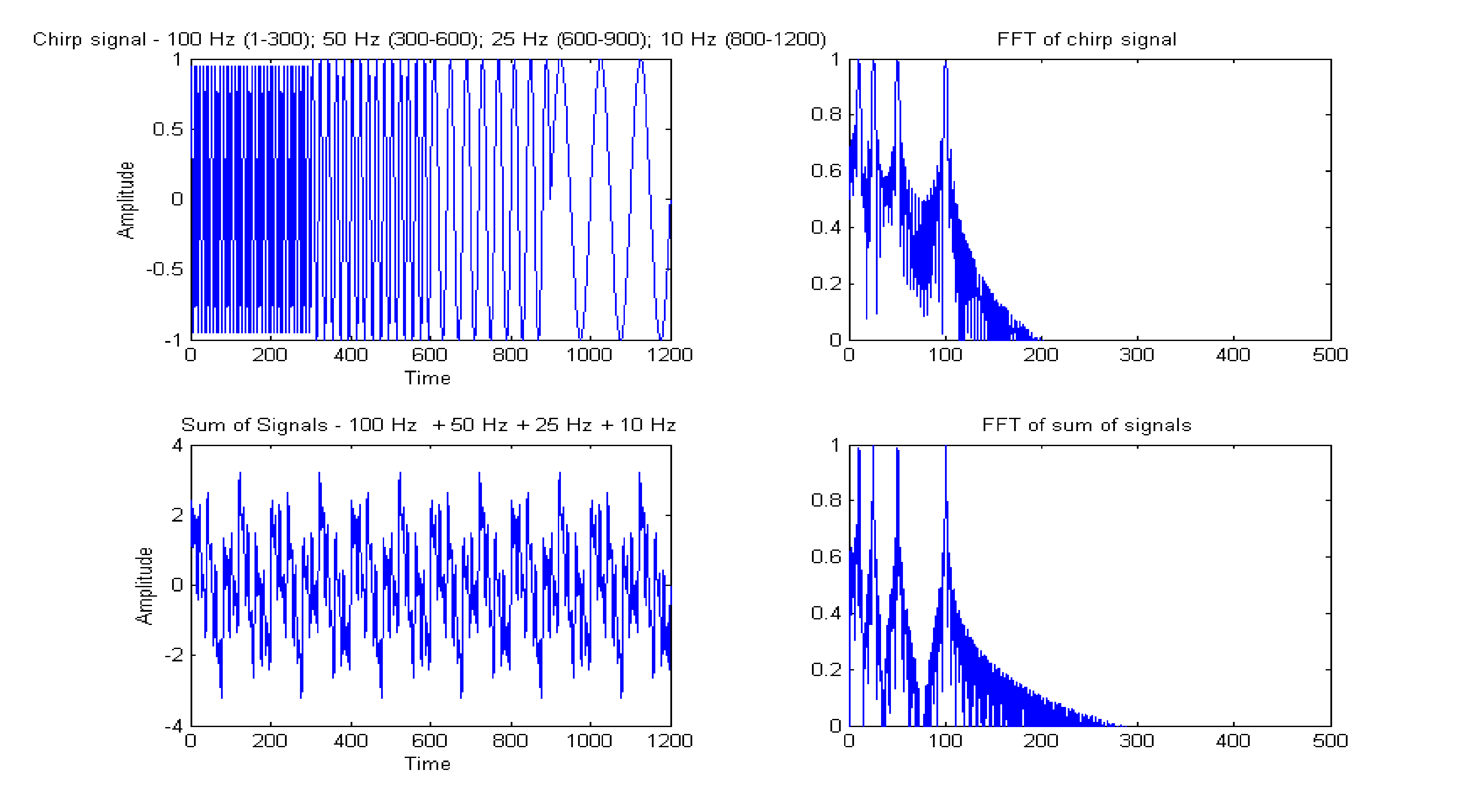
Fourier Analysis: Strengths and Weaknesses

- **Consider a discrete signal and its DTFT:**

$$x[n] \Leftrightarrow X(e^{jw}) = \sum_n x[n] e^{-jwn}$$

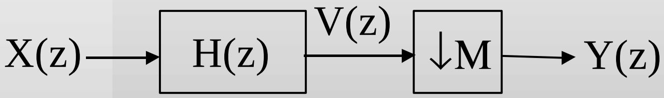
- **Strengths:**
  - **analyzing stationary signals, i.e. signals that do not involve abrupt changes -- periodic and quasi-periodic signals**
  - **good frequency localization**
- **Weaknesses**
  - **not good for non-stationary signals**
  - **poor time localization**

Fourier Analysis: Strengths and Weaknesses (2)



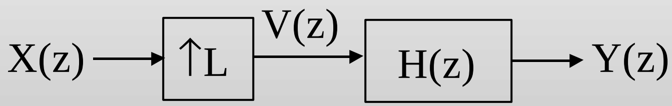
Multirate System Basics

Decimation



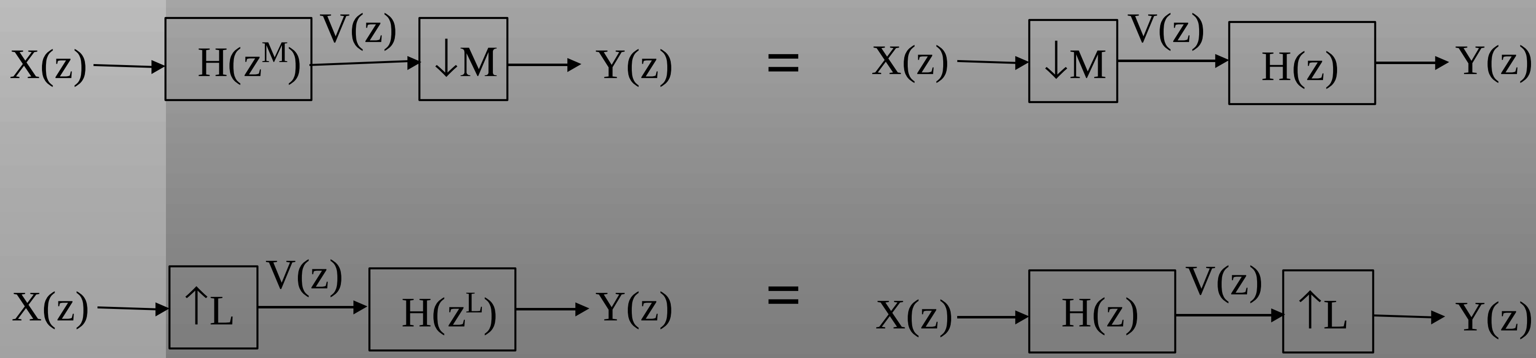
$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right)$$

Interpolation



$$Y(z) = X\left(z^L\right)$$

Noble Identities



Multirate System Basics: Polyphase Structure

- **Suppose we have a signal of length 16 and want to divide it into 4 groups (M = 4)**

$$X(z) = x[0] + x[1]z^{-1} + \ldots + x[15]z^{-15}$$

$$= x[0] + x[4]z^{-4} + x[8]z^{-8} + x[12]z^{-12}$$

$$+ z^{-1}\left\{x[1] + x[5]z^{-4} + x[9]z^{-8} + x[13]z^{-12}\right\}$$

$$+ z^{-2}\left\{x[2] + x[6]z^{-4} + x[10]z^{-8} + x[14]z^{-12}\right\}$$

$$+ z^{-3}\left\{x[3] + x[7]z^{-4} + x[11]z^{-8} + x[15]z^{-12}\right\}$$

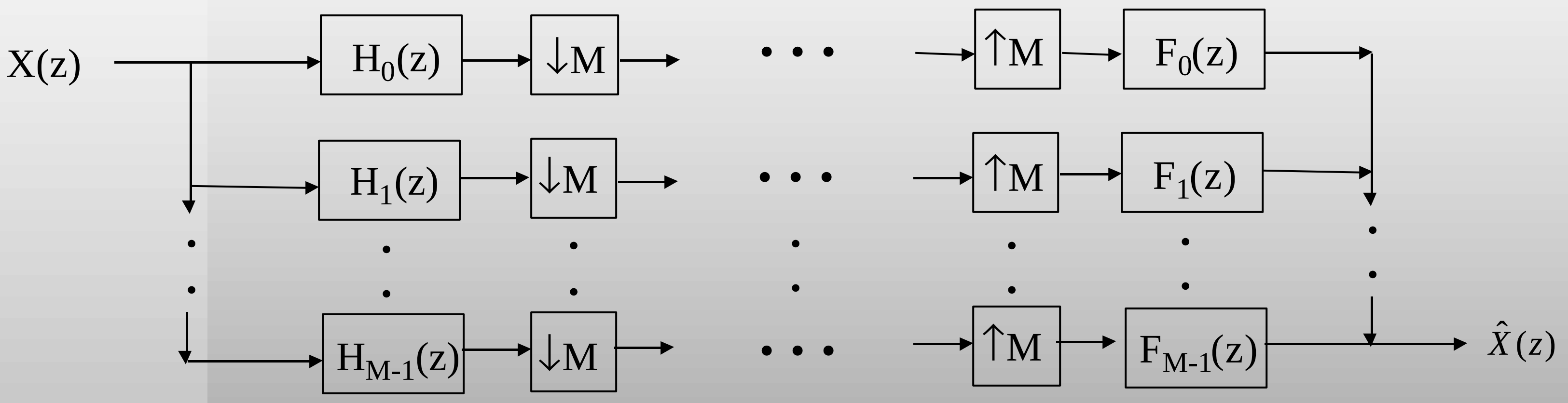
$$= \sum_{l=0}^{3} z^{-l} \sum_{n} x[4n + l]z^{-4n}$$

- **In general:**

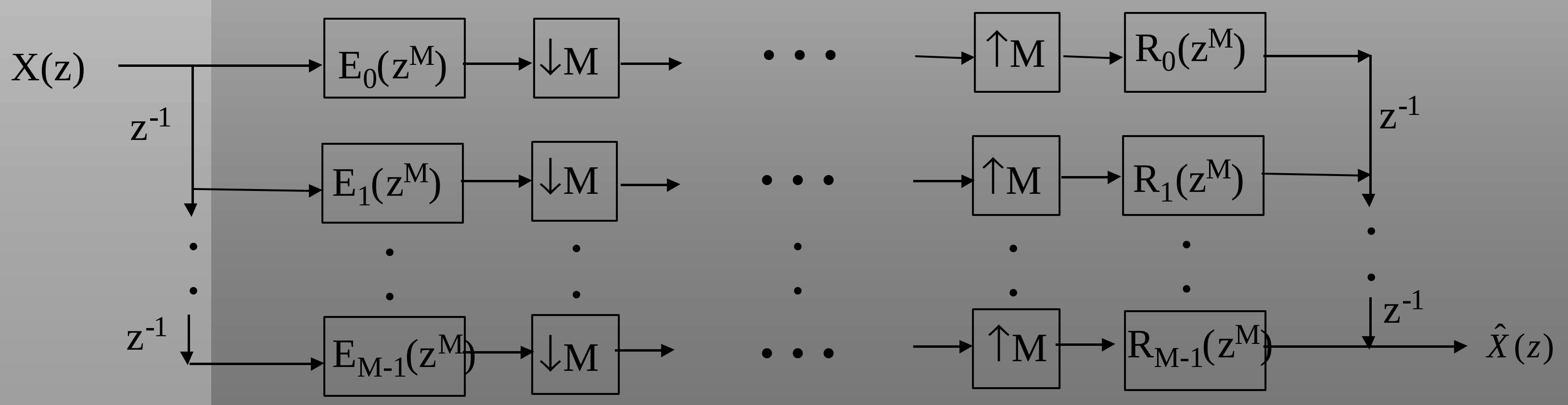
$$X(z) = \sum_{l=0}^{M-1} z^{-l} \sum_{n} x[nM + l]z^{-nM} = \sum_{l=0}^{M-1} z^{-l} E_l\left(z^M\right) \qquad \text{Type-I}$$

$$X(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} \sum_{n} x[nM + M - 1 - l]z^{-nM} = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_l\left(z^M\right) \qquad \text{Type-II}$$

Basic Filter Bank Structure

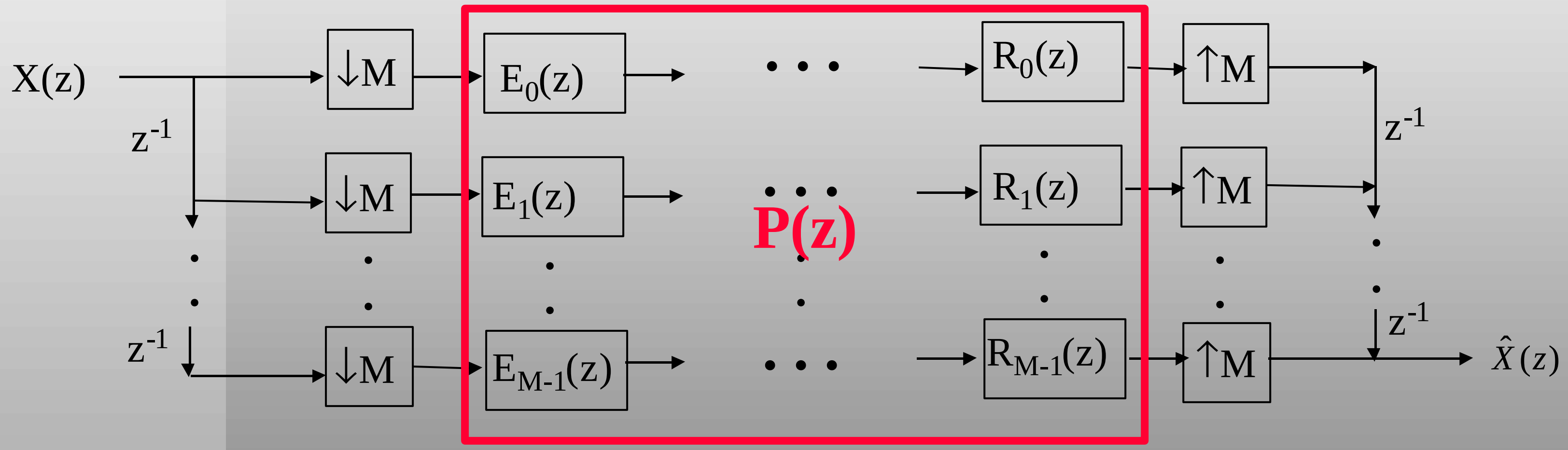


Polyphase Representation



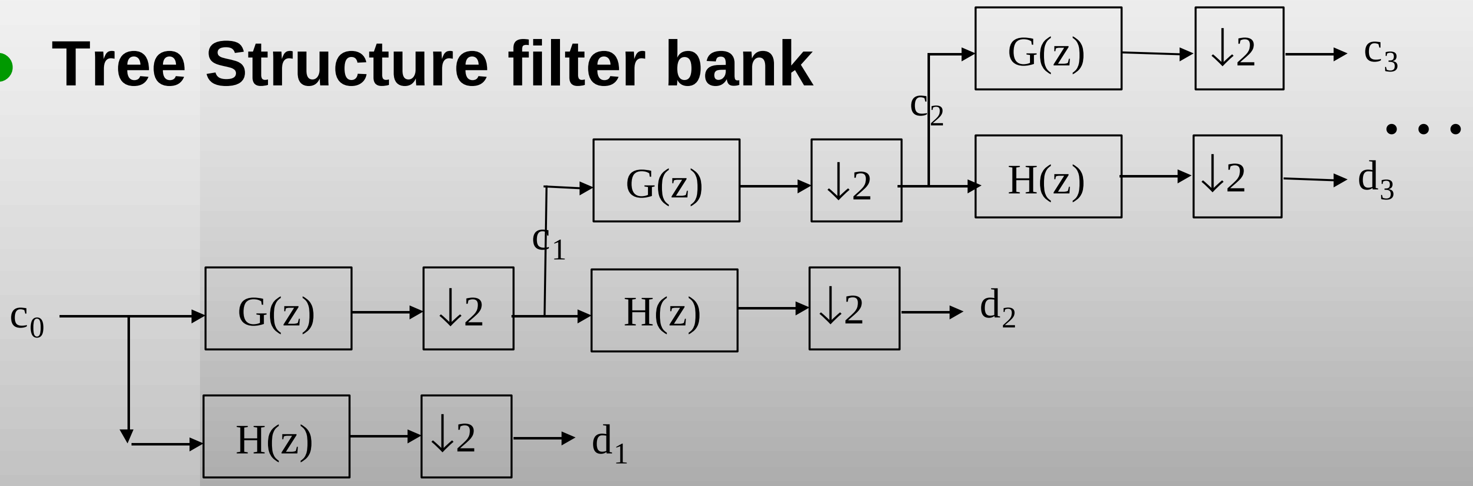
Basic Filter Bank Structure (2)

Polyphase Representation using Noble Identities

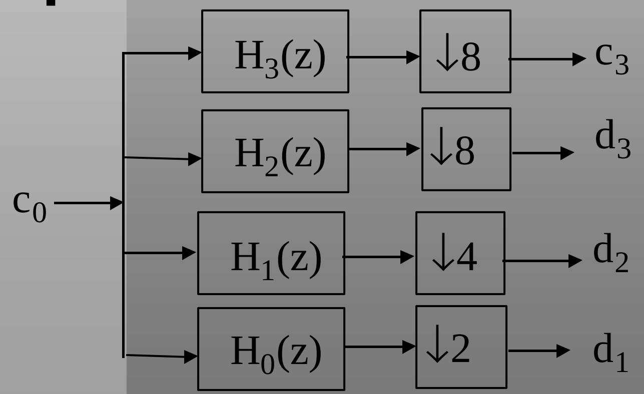


Other Filter Bank Structure

- **Tree Structure filter bank**



- **Equivalent to**



$H_3(z) = G(z)G(z^2)G(z^4)$

$H_2(z) = G(z)G(z^2)H(z^4)$

$H_1(z) = G(z)H(z^2)$

$H_0(z) = H(z)$

Perfect Reconstruction

- **Condition for PR in MR system:**
  - **no amplitude distortion, i.e. overall system response can be multiplied by a constant *c***
  - **no phase distortion, i.e. overall system response is linear phase, not necessarily individual filters**
  - **no aliasing distortion**
- **Conclusion:**

$$\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z) = cz^{-K}\mathbf{I} \Rightarrow \mathbf{R}(z) = cz^{-K}\mathbf{E}^{-1}(z) = cz^{-K}\frac{Adj\{\mathbf{E}(z)\}}{\det\{\mathbf{E}(z)\}}$$

or more general P(z) is a pseudo-circulant matrix

$$\mathbf{P}(z) = \begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix} \quad \mathbf{P}(z) = \begin{bmatrix} \mathbf{0} & \mathbf{I_{M-r}} \\ \mathbf{z^{-1}I}_r & \mathbf{0} \end{bmatrix}, \quad 0 \le r \le M-1 \text{ for PR}$$

Perfect Reconstruction (2)

- **So for $\underline{\mathbf{E}}(z)$ to be FIR (implies H(z) is FIR), $\underline{\mathbf{R}}(z)$ is going to be IIR (have possibilities of being unstable) unless *det* {$\underline{\mathbf{E}}(z)$} = cz^{-K}**
- **Paraunitary condition for PR**

$$\tilde{\mathbf{E}}(z)\mathbf{E}(z) = c\mathbf{I} \Rightarrow \mathbf{E}^{-1}(z) = \frac{1}{c}\tilde{\mathbf{E}}(z) \Rightarrow \mathbf{R}(z) = cz^{-K}\tilde{\mathbf{E}}(z), \qquad \tilde{\mathbf{E}}(z) = \mathbf{E}_*^T\left(1/z^*\right)$$

- **For M-band system, the filter requirements are:**

$$f_k[n] = ch_k^*[L-n] \Leftrightarrow F_k(z) = cz^{-L}\tilde{H}_k(z), \qquad L = Mk + M + 1, \;\; 0 \le k \le M-1$$

$$\sum_{k=0}^{M-1}\left|H_k(e^{jw})\right|^2 = c \qquad power \;\; complementary$$

$$\tilde{H}(z)H_k(z) \;\; is \;\; an \;\; M^{th} band \;\; filter$$

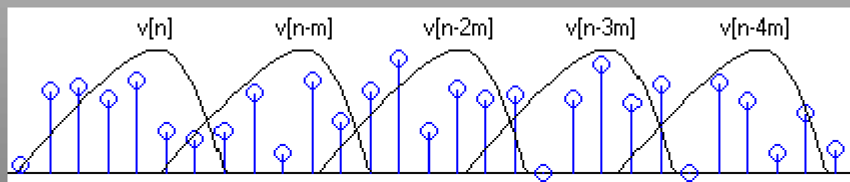
STFT: Transform Interpretation

- **Consider a discrete signal and its STFT:**

$$X_{STFT}(m, e^{jw_k}) = \sum_n x[n]v^*[n - Nm]e^{-jw_k n}; \qquad w_k = \frac{2p}{M}k$$

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{M-1} X_{STFT}(m, e^{jw_k})v(n - Nm)e^{jw_k n}$$

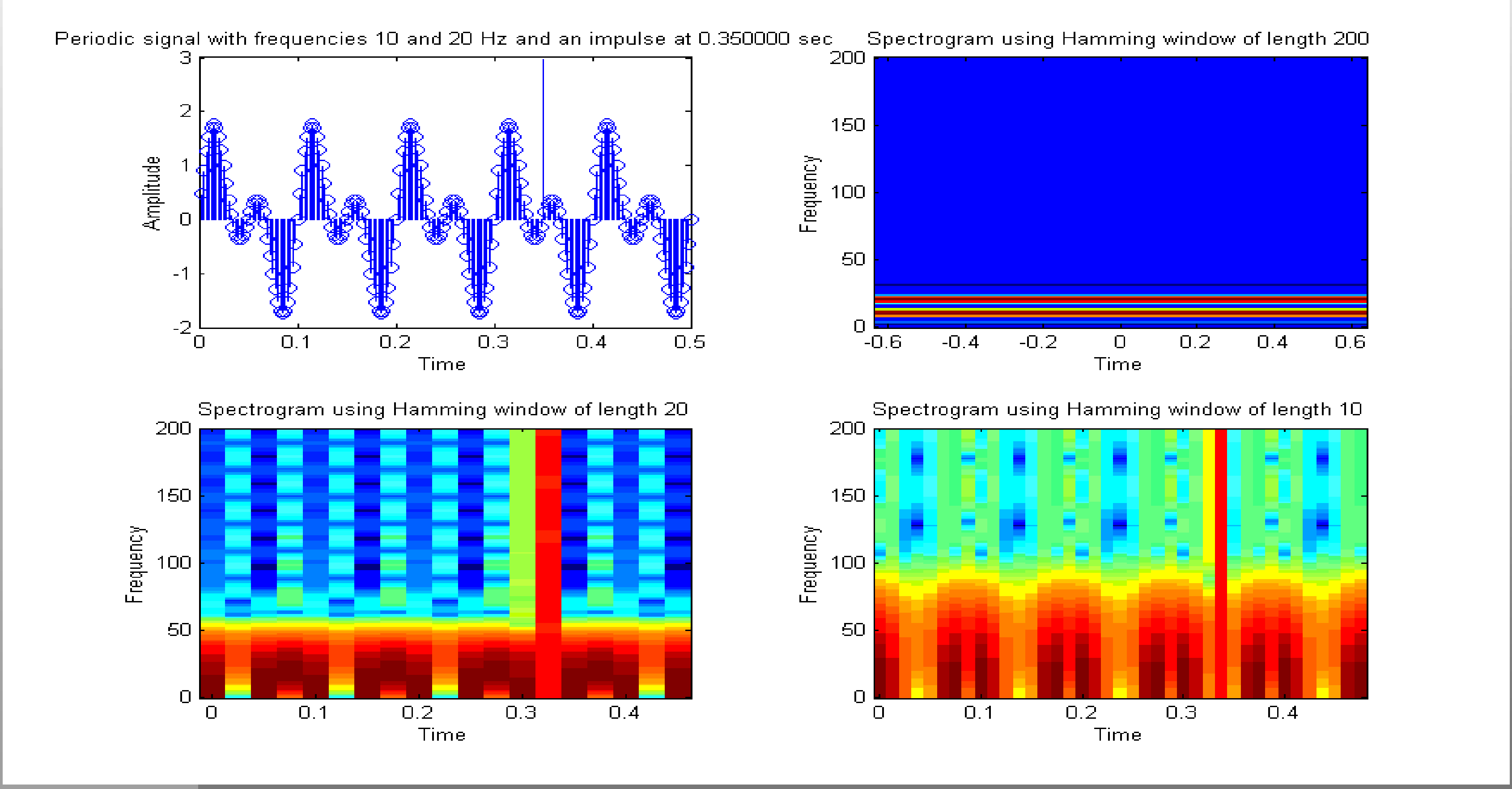
- $V(e^{jw_k}) = \sum_n v[n]e^{jw_k n}; \quad V(e^{j0}) = \sum_n v[n] \neq 0$ **==> the modulated window is similar to a lowpass filter**



- **Applying window *v[n]* shifted in time to get some time localization**

STFT: Transform Interpretation (2)

- **Problem: window width is constant throughout the transform**



Wavelets: Transform Interpretation

- **Project signals onto another set of basis functions which are dilated and shifted version of a prototype function. This prototype function is known as the mother wavelet.**
- **No longer dealing with time-frequency analysis, but rather time-scale. Scale is proportional to inverse of frequency**
- **Continuous Wavelet Transform:**

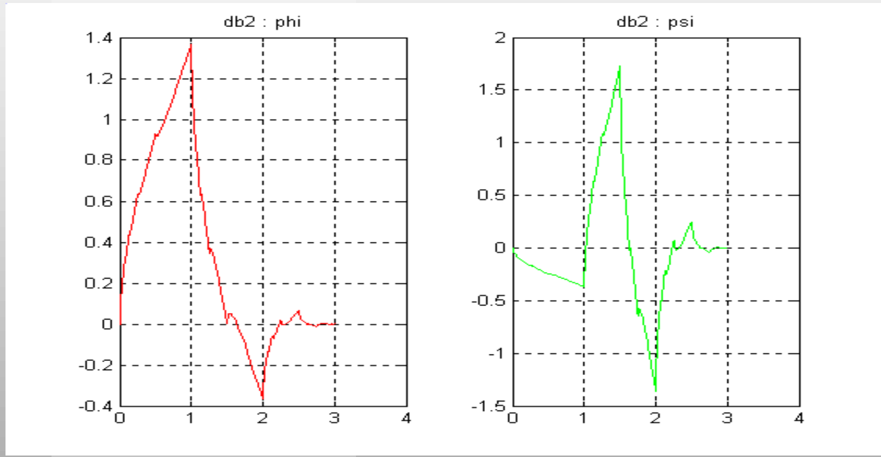
$$X_{CWT}(b,a) = \frac{1}{\sqrt{a}} \int_t x(t) y^*\left(\frac{t-b}{a}\right) dt$$

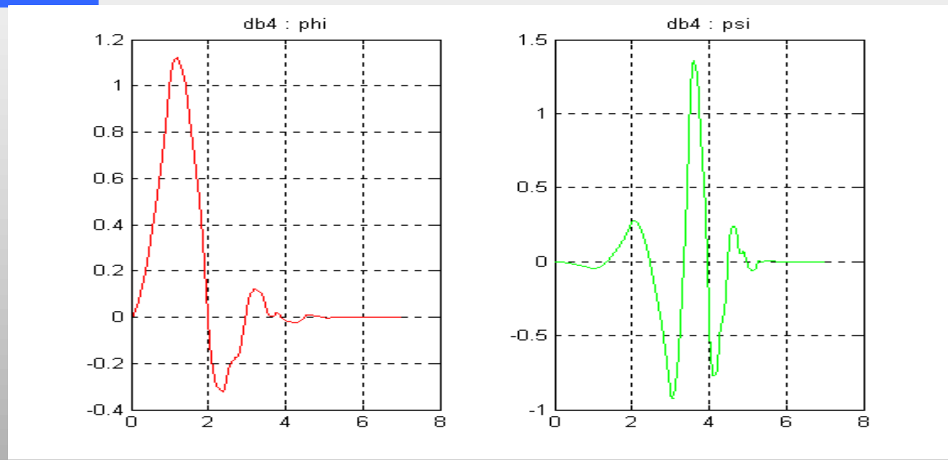
$$\hat{x}(t) = \frac{1}{C_y} \int_b \int_a \frac{1}{a^2} X_{CWT}(b,a) y\left(\frac{t-b}{a}\right) da \, db,$$

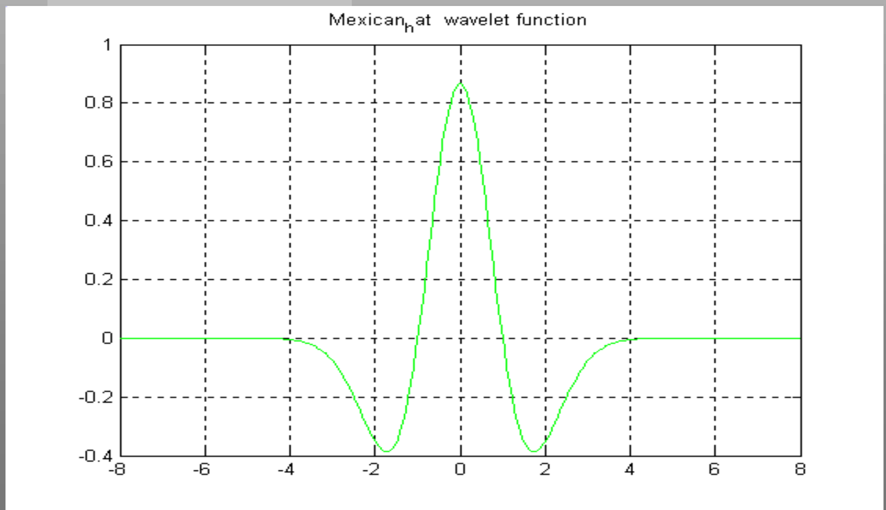
$$C_y = \int_\Omega \frac{|\Psi(j\Omega)|^2}{|\Omega|} d\Omega < \infty, \qquad \Psi(0) = \int_t y(t) dt = 0$$

- **Some wavelets: Daubechies 2, 4, Mexican Hat, Morlet, ...**

Some wavelets







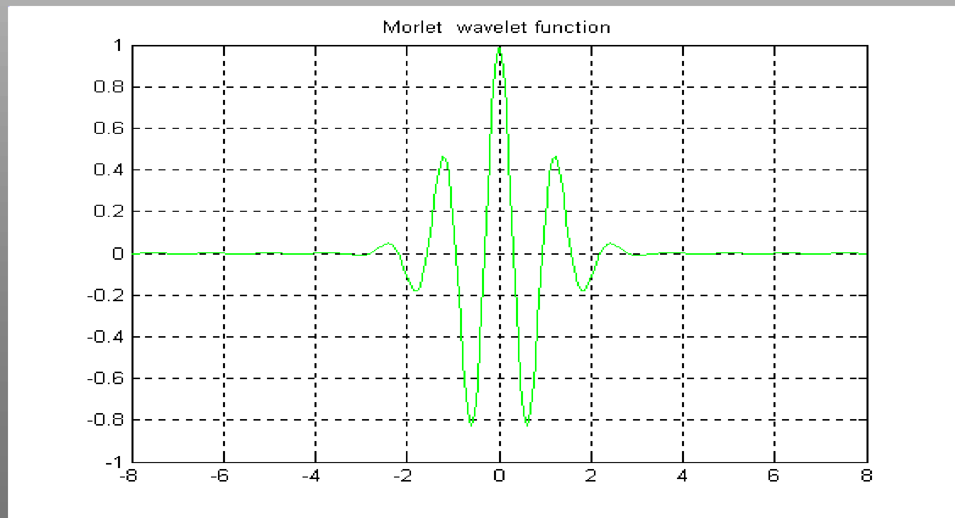
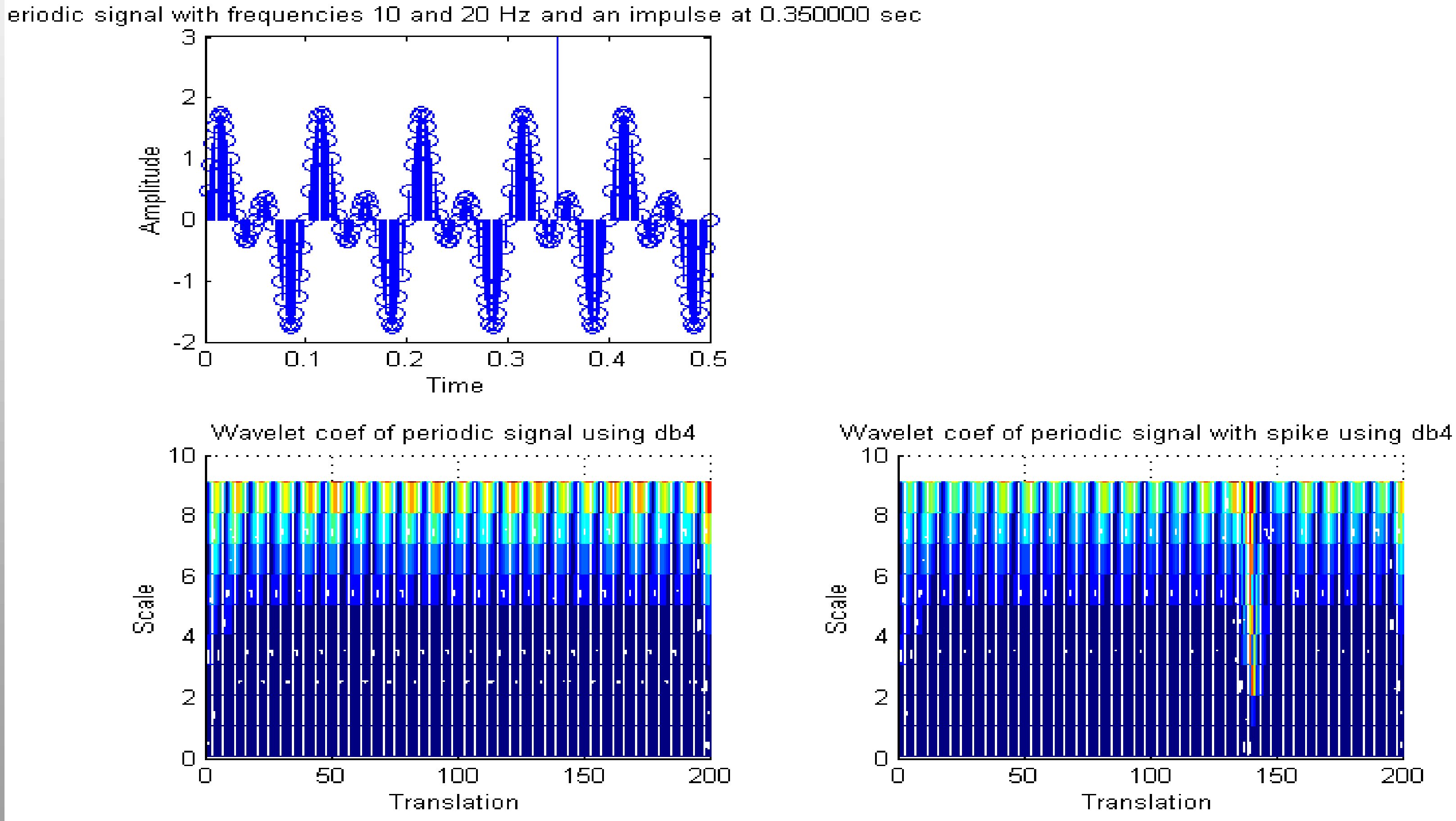


Illustration of wavelet coefficients



Wavelets: Transform Interpretation (2)

- **Discrete Wavelet Transform (still continuous time)**

$$X_{DWT}(a^k nT, a^k) = a^{-k/2}\int_t x(t)\tilde{y}^*\left(\frac{t-a^k nT}{a^k}\right)dt \qquad \tilde{y}_{kn}(t) = a^{-k/2}\tilde{y}(a^{-k}(t-a^k nT))$$

$$\hat{x}(t) = \sum_k \sum_n X_{DWT}(a^k nT, a^k) a^{-k/2} y\left(\frac{t-a^k nT}{a^k}\right), \qquad y_{kn}(t) = a^{-k/2} y(a^{-k}(t-a^k nT))$$

biorthogonal condition: $\langle y_{mn}, \tilde{y}_{lk}\rangle = d_{ml} d_{nk}, \quad m,n,l,k \in Z$

orthonormal condition: $\langle y_{mn}, y_{lk}\rangle = d_{ml} d_{nk}, \quad m,n,l,k \in Z$

- **if a = 2, it's called dyadic sampling ==> shifting and dilating the mother wavelet by a power of 2. It can be shown this is complete (covers the whole signal space) and PR is possible.**

Wavelets: Transform Interpretation (3)

- **corresponds to multiresolution analysis and synthesis. Let T=1, the biorthogonal wavelet pair is:**

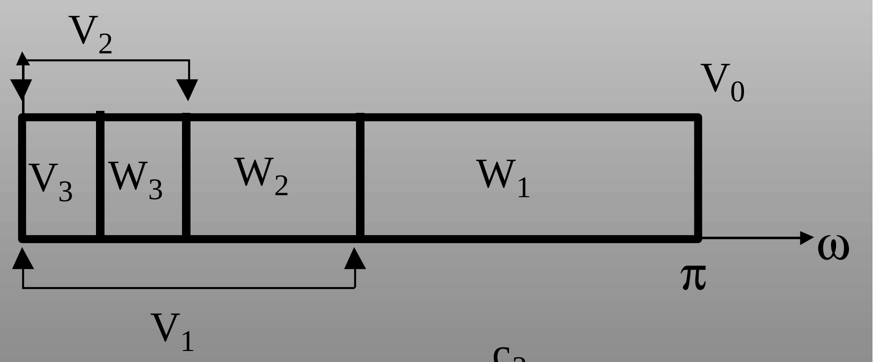
$$\phi_{kn}(t) = 2^{-\frac{k}{2}}\phi(2^{-k}t-n), \qquad V_k = span\{\phi(2^{-k}t-n), n\in Z\}$$

V_k is the scaling subspace

$$\psi_{kn}(t) = 2^{-\frac{k}{2}}\psi(2^{-k}t-n), \qquad W_k = span\{\psi(2^{-k}t-n), n\in Z\}$$

W_k is the wavelet subspace

$$\tilde{\psi}_{kn}(t) = 2^{-\frac{k}{2}}\tilde{\psi}(2^{-k}t-n)$$



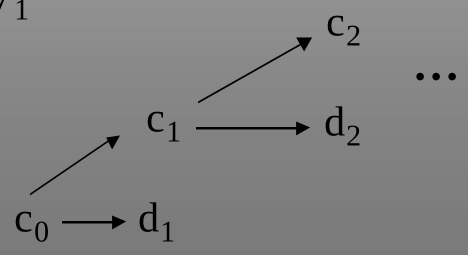
$$V_k = V_{k+1}\oplus W_{k+1} = \dots W_{k+2}\oplus W_{k+1}\oplus W_k\oplus W_{k-1}\oplus W_{k-2}\dots$$

$$x(t) = \sum_k\sum_n \langle x,\tilde{\psi}_{kn}\rangle\psi_{kn}(t) = \sum_k\sum_n d_k[n]\psi_{kn}(t)$$

$$d_k[n] = \langle x,\tilde{\psi}_{kn}\rangle \qquad c_k[n] = \langle x,\phi_{kn}\rangle$$

$$y_{k+1}(t) = \sum_n d_{k+1,n}\psi_{k+1,n}(t) \qquad x_{k+1}(t) = \sum_n c_{k+1,n}\phi_{k+1,n}(t)$$

$$x_k = x_{k+1} + y_{k+1} \Rightarrow c_k = c_{k+1} + d_{k+1}$$



Wavelets: Transform Interpretation (4)

- **Analysis by Multirate Filtering**

$$c_{k+1}[l] = \sum_n c_k[n] h_0[2l-n] \qquad d_{k+1}[l] = \sum c_k[n] h_1[2l-n], \qquad l \in Z$$

- **Synthesis by Multirate Filtering**

$$c_k[n] = \sum c_{k+1}[l] g_0[n-2l] + d_{k+1}[l] g_1[n-2l]$$

- **Relation to Filter Bank**
  - **since $V_k = V_{k+1} \oplus W_{k+1}$, there exists two sequences *$h_0[l]$* and *$h_1[l]$* such that**

$$f_{0n}(t) = \sum_l h_0[2l-n] f_{1l}(t) + h_1[2l-n] y_{1l}(t) \qquad \textit{Decomposition}$$

$$f_{1l}(t) = \sum_n g_0[n-2l] f_{0n}(t) \qquad \textit{Synthesis}$$

$$y_{1l}(t) = \sum_n g_1[n-2l] f_{0n}(t)$$

Wavelets: Transform Interpretation (4)

- **If we multiply the decomposition and synthesis equation by $\tilde{f}_{1l}(t)$ and $\tilde{y}_{1l}(t)$ we get:**

$$h_0[2l-n] = \langle f_{0n}, \tilde{f}_{1l} \rangle \qquad h_1[2l-n] = \langle f_{0n}, \tilde{y}_{1l} \rangle$$

$$g_o[n] = \langle f_{10}, \tilde{f}_{0n} \rangle \qquad g_1[n] = \langle y_{10}, \tilde{f}_{0n} \rangle$$

and after some manipulation, we get:

$$\sum_n g_0[n] h_0[2l-n] = d_{l0} \Leftrightarrow G_0(z)H_0(z) + G_0(-z)H_0(-z) = 2$$

$$\sum_n g_1[n] h_1[2l-n] = d_{l0} \Leftrightarrow G_1(z)H_1(z) + G_1(-z)H_1(-z) = 2$$

$$\sum_n g_0[n] h_1[2l-n] = 0 \Leftrightarrow G_0(z)H_1(z) + G_0(-z)H_1(-z) = 0$$

$$\sum_n g_1[n] h_0[2l-n] = 0 \Leftrightarrow G_1(z)H_0(z) + G_1(-z)H_0(-z) = 0$$

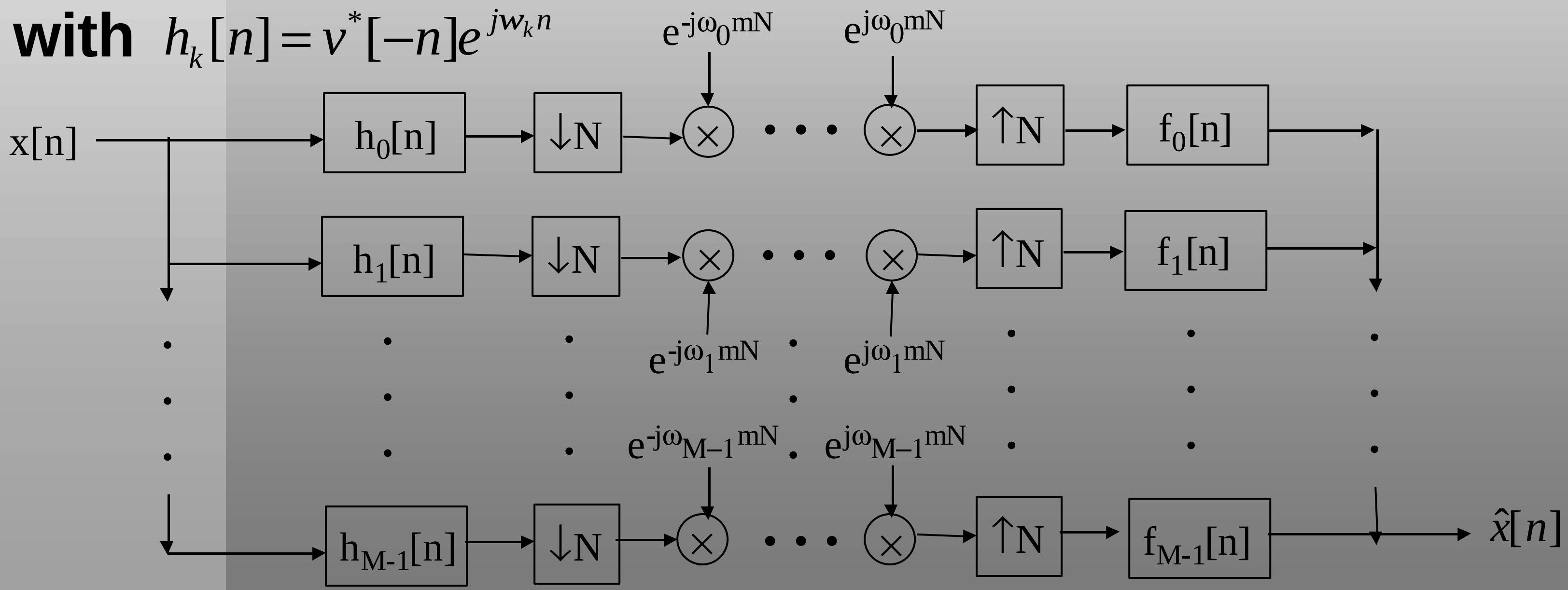
This is equivalent to PR condition of a 2-channel filter bank

STFT (filter bank interpretation)

- **Can interpret the transform as convolving the signal x[n] by a bank of bandpass filters *$h_k[n]$*:**

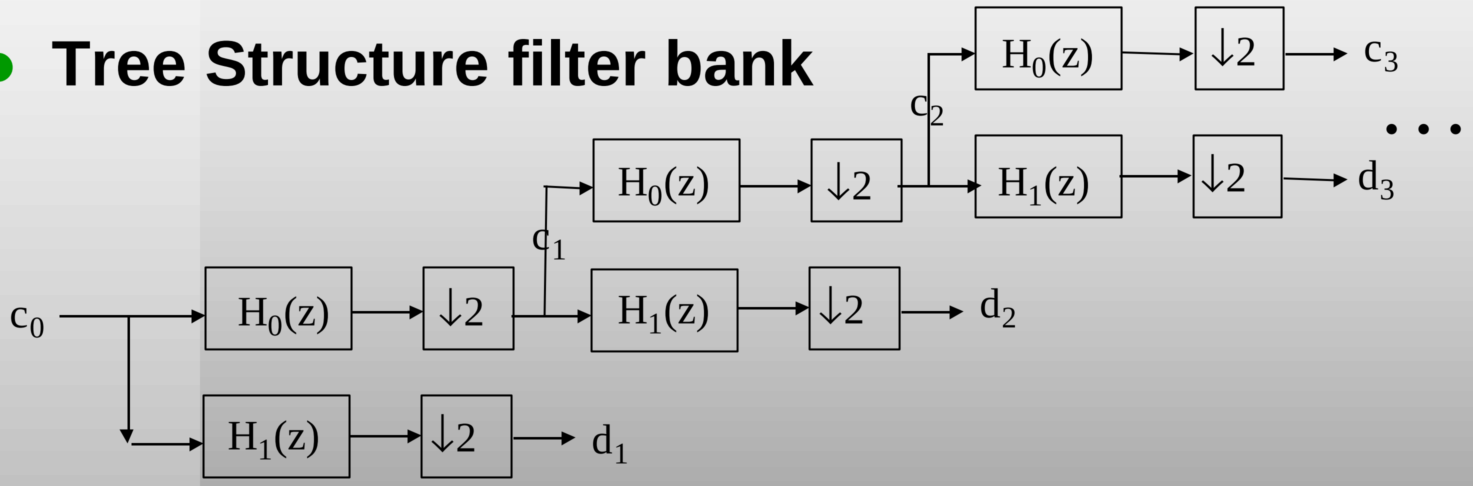
$$X_{STFT}(m,e^{jw_k}) = e^{-jw_k mN} \sum_n x[n] v^*[-(Nm-n)] e^{-jw_k(-(Nm-n))}$$

with $h_k[n] = v^*[-n] e^{jw_k n}$

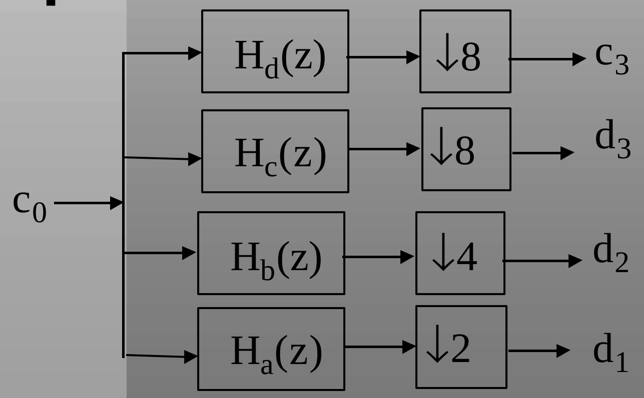


DWT (filter bank interpretation)

- **Tree Structure filter bank**



- **Equivalent to**



$H_d(z) = H_0(z)\,H_0(z^2)\,H_0(z^4)$

$H_c(z) = H_0(z)H_0(z^2)H_1(z^4)$

$H_b(z) = H_0(z)H_1(z^2)$

$H_a(z) = H_1(z)$

Future Topics

- **Construction of wavelets from filter bank iteration, splines, ...**
- **Some properties of wavelets: k-Regularity, Moments**
- **Fast algorithm for computing DTWT (Discrete Time Wavelet Transform)**
- **Multidimensional multirate system**
- **Application - Pattern Recognition: Edge detection (Modulus Maxima), Segmentation, Registration, …**
- **And many more**

References

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