Estimation of 2-D Frequencies Using Modified Matrix Pencil Method

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Applications

- Communications
- Sonar
- Radar
- Synthetic aperture radar imaging
- Nuclear magnetic resonance imaging
- Ultrasound imaging

Problem

$$x(m,n) = \underbrace{\sum_{k=1}^{N_s} \alpha_k \exp\{j2\pi f_{1k} m + j2\pi f_{2k} n + j\phi_k\}}_{s(m,n)} + \eta(m,n),$$

$$0 \le m \le M-1,$$
$$0 \le n \le N-1$$

α_k, f_{1k}, f_{2k}, ϕ_k → **Unknown**

- Assume N_s is known
- Goal: Estimate only f_{1k} and f_{2k}

Some Past Approaches

- Nonparametric
  - 2-D Periodogram
  - 2-D Correlogram
- Parametric
  - 2-D Autoregressive
  - 2-D Maximum Entropy

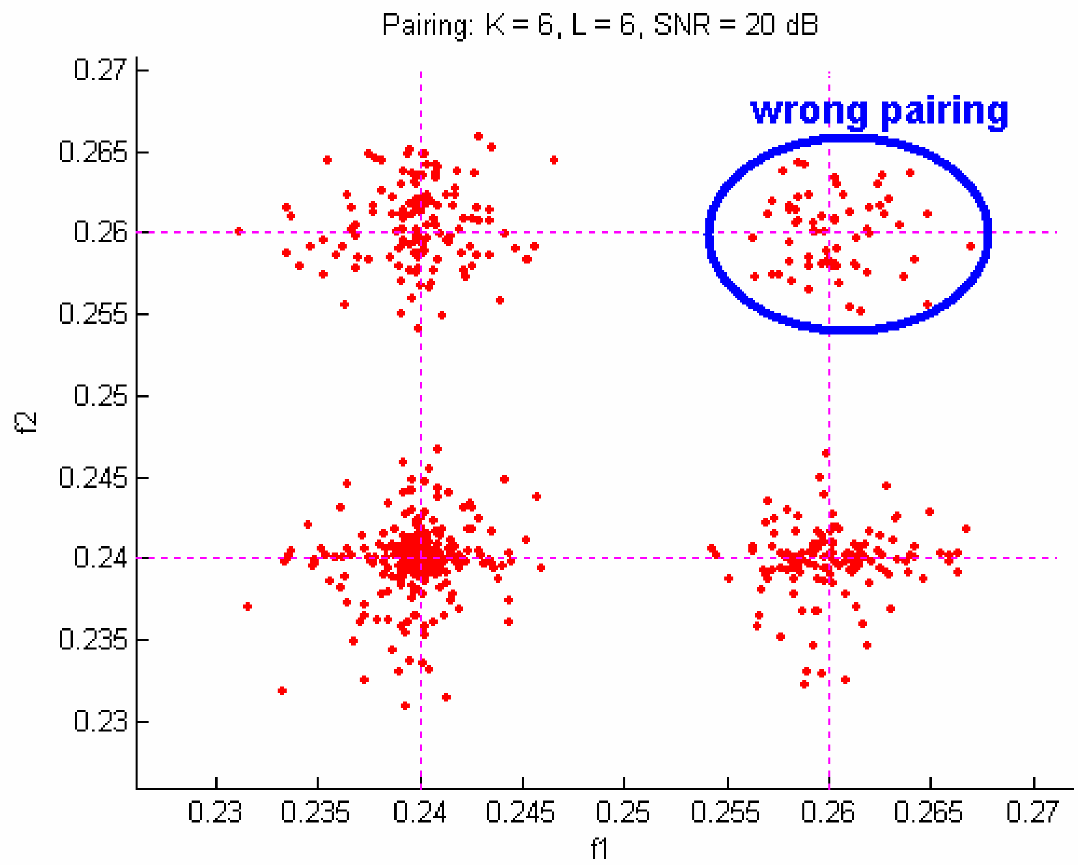
Small data size → limit resolution

Search spectral peaks in 2-D space → computationally expensive

Matrix Enhancement and Matrix Pencil Method (MEMP)

- Developed by Y. Hua [1992]
- Accurate estimate
  - Estimate frequencies individually
  - Followed by pairing operation
- Problem: pairing
  - Add computational complexity
  - Does not provide accurate and consistent pairing
  - Searching operation

MEMP Pairing Problem



(f1,f2) = (0.26,0.24)

(f1,f2) = (0.24,0.24)

(f1,f2) = (0.24,0.26)

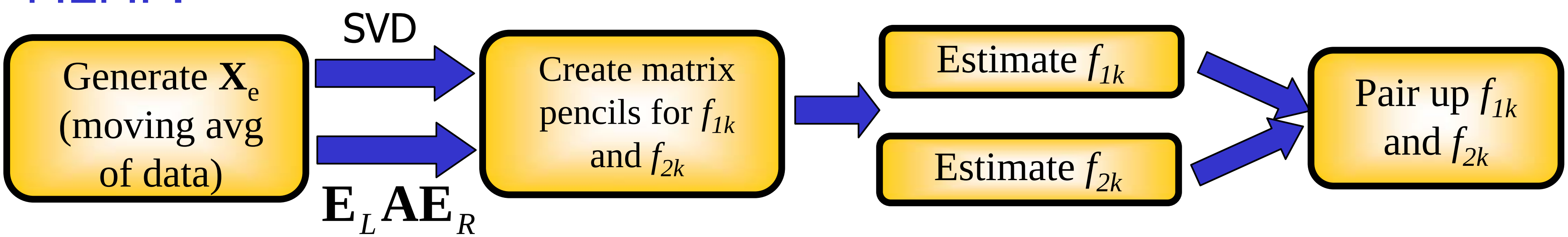
200 independent estimates, M=N=20, K=L=6, SNR = 20 dB

Modified Matrix Enhancement and Matrix Pencil Method (MMEMP)

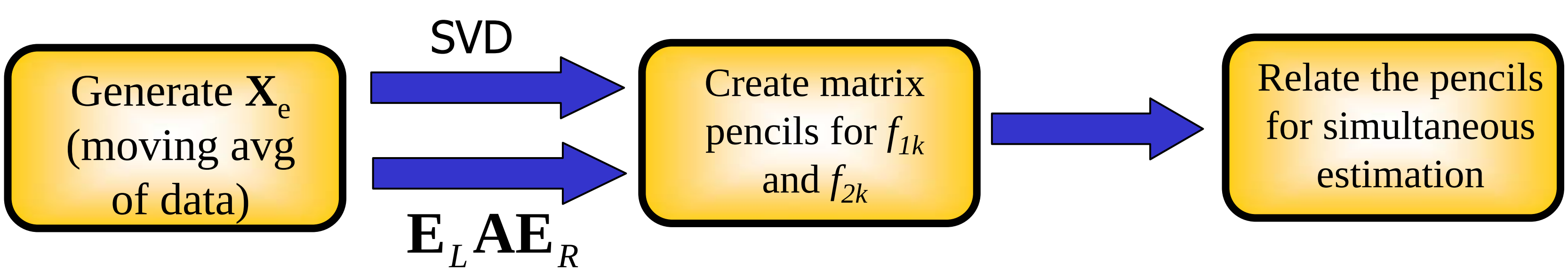
- Estimate frequencies simultaneously
  - No pairing
  - Estimation as accurate as MEMP
  - Provide accurate and consistent pairing results

MEMP vs. MMEMP -- Basic Idea

MEMP:



MMEMP:



Strategy for Simultaneous Estimation

- Obtain matrix pencil expression for f_{1k}
  - Obtain eigenvector **W**
- Obtain matrix pencil expression for f_{2k}
  - Obtain eigenvector **R**
- Relate **W** and **R**
- Using **W** to simultaneously obtain f_{1k} and f_{2k} without losing accuracy compared to estimating them separately

MMEMP Formulation

$$x(m,n)=\sum_{k=1}^{N_s}\alpha_k \exp\{j\phi + j2\pi f_{1k}m + j2\pi f_{2k}n\} + \eta(m,n)$$

$\exp\{\bullet\}$ $\exp\{\bullet\}$

$$x(m,n)=\sum_{k=1}^{N_s} a_k y_k^m z_k^n + \eta(m,n)$$

Enhancement Matrix

$$x(m,n)=\sum_{k=1}^{N_s} a_k y_k^m z_k^n + \eta(m,n)$$

$$\mathbf{X}_e = \underbrace{\begin{bmatrix} \mathbf{X}_0 & \mathbf{X}_1 & \cdots & \mathbf{X}_{M-K} \\ \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_{M-K+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_{K-1} & \mathbf{X}_K & \cdots & \mathbf{X}_{M-1} \end{bmatrix}}_{(M-K+1)(N-L+1)} \Big\} KL,$$

$$\text{where } \mathbf{X}_m = \begin{bmatrix} x(m,0) & x(m,1) & \cdots & x(m,N-L) \\ x(m,1) & x(m,2) & \cdots & x(m,N-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ x(m,L-1) & x(m,L) & \cdots & x(m,N-1) \end{bmatrix}$$

Decomposition of $\mathbf{X}_e$

$$x(m,n)=\sum_{k=1}^{N_s} a_k y_k^m z_k^m + \eta(m,n)$$

$$\mathbf{X}_e = \begin{bmatrix} \mathbf{X}_0 & \mathbf{X}_1 & \cdots & \mathbf{X}_{M-K} \\ \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_{M-K+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_{K-1} & \mathbf{X}_K & \cdots & \mathbf{X}_{M-1} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{X}_e = \mathbf{E}_L \mathbf{A} \mathbf{E}_R$$

$$\mathbf{E}_L \equiv \begin{bmatrix} \mathbf{Z}_L \\ \mathbf{Z}_L \mathbf{Y}_d \\ \vdots \\ \mathbf{Z}_L \mathbf{Y}_d^{K-1} \end{bmatrix}, \quad \mathbf{E}_R \equiv \begin{bmatrix} \mathbf{Z}_R & \mathbf{Y}_d \mathbf{Z}_R & \cdots & \mathbf{Y}_d^{M-K} \mathbf{Z}_R \end{bmatrix}, \quad \mathbf{Z}_L \equiv \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_{N_s} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{L-1} & z_2^{L-1} & \cdots & z_{N_s}^{L-1} \end{bmatrix}, \quad \mathbf{Z}_R \equiv \begin{bmatrix} 1 & z_1 & \cdots & z_1^{N-1} \\ 1 & z_2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_{N_s} & \cdots & z_{N_s}^{N-1} \end{bmatrix},$$

$$\mathbf{A} \equiv \mathrm{diag}(a_1, a_2, \ldots, a_{N_s})$$

$$\mathbf{Y}_d = \mathrm{diag}(y_1, y_2, \ldots, y_{N_s}) \Leftrightarrow f_{1k}$$

Decomposition of $\mathbf{X}_e$

$$x(m,n)=\sum_{k=1}^{N_s} a_k y_k^m z_k^m + \eta(m,n)$$

$$\mathbf{X}_e = \mathbf{E}_L \mathbf{A} \mathbf{E}_R = \mathbf{X}_e = \sum_{k=1}^{J} \sigma_k \mathbf{u}_k \mathbf{v}_k = \mathbf{U}\Sigma\mathbf{V}^H = \begin{bmatrix}\mathbf{U}_s & \mathbf{U}_n\end{bmatrix}\begin{bmatrix}\Sigma_s & \mathbf{0} \\ \mathbf{0} & \Sigma_n\end{bmatrix}\begin{bmatrix}\mathbf{V}_s^H \\ \mathbf{V}_n^H\end{bmatrix}$$

$$J = \min\left(KL, (M-K+1)(N-L+1)\right)$$

Signal and Noise Subspace

$$
\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{N_s} & \mathbf{u}_{N_s+1} & \mathbf{u}_{N_s+2} & \cdots & \mathbf{u}_J \end{bmatrix}
\begin{bmatrix} \sigma_1 & 0 & & \cdots & & & 0 \\ 0 & \sigma_2 & & & & & \\ & & \ddots & & & & \vdots \\ & & & \sigma_{N_s} & & & \\ \vdots & & & & \sigma_{N_s+1} & & \\ & & & & & \sigma_{N_s+2} & \\ & & & & & \ddots & 0 \\ 0 & & \cdots & & & 0 & \sigma_J \end{bmatrix}
\begin{bmatrix} \mathbf{v}_1^H \\ \mathbf{v}_2^H \\ \vdots \\ \mathbf{v}_{N_s}^H \\ \mathbf{v}_{N_s+1}^H \\ \mathbf{v}_{N_s+2}^H \\ \vdots \\ \mathbf{v}_J^H \end{bmatrix}
$$

$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{N_s}$: Signal Subspace

$\mathbf{u}_{N_s+1}, \mathbf{u}_{N_s+2}, \cdots, \mathbf{u}_J$: Noise Subspace

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{N_s} \geq \sigma_{N_s+1} \geq \sigma_{N_s+2} \geq \cdots \geq \sigma_J$$

$$\mathbf{U}_s \perp \mathbf{U}_n$$

Orthogonality of Subspace

$$\mathbf{X}_e = \mathbf{E}_L \mathbf{A} \mathbf{E}_R = \mathbf{U}_s \Sigma_s \mathbf{V}_s^H + \mathbf{U}_n \Sigma_n \mathbf{V}_n^H$$

- rank($\mathbf{X}_e$) = N_s iff rank($\mathbf{E}_L$) = N_s and rank($\mathbf{E}_R$) = N_s
  - rank($\mathbf{E}_L$) = N_s if K $\geqslant$ N_s and L $\geqslant$ N_s
  - rank($\mathbf{E}_R$) = N_s if M-K+1 $\geqslant$ N_s and N-L+1 $\geqslant$ N_s
- range($\mathbf{X}_e$) = range($\mathbf{E}_L$) = range($\mathbf{U}_s$)

$$\mathbf{U}_s \perp \mathbf{U}_n \Rightarrow \mathbf{E}_L \perp \mathbf{U}_n$$

Matrix Pencil for f_1

$$\mathbf{U}_s = \mathbf{E}_L \mathbf{T}$$

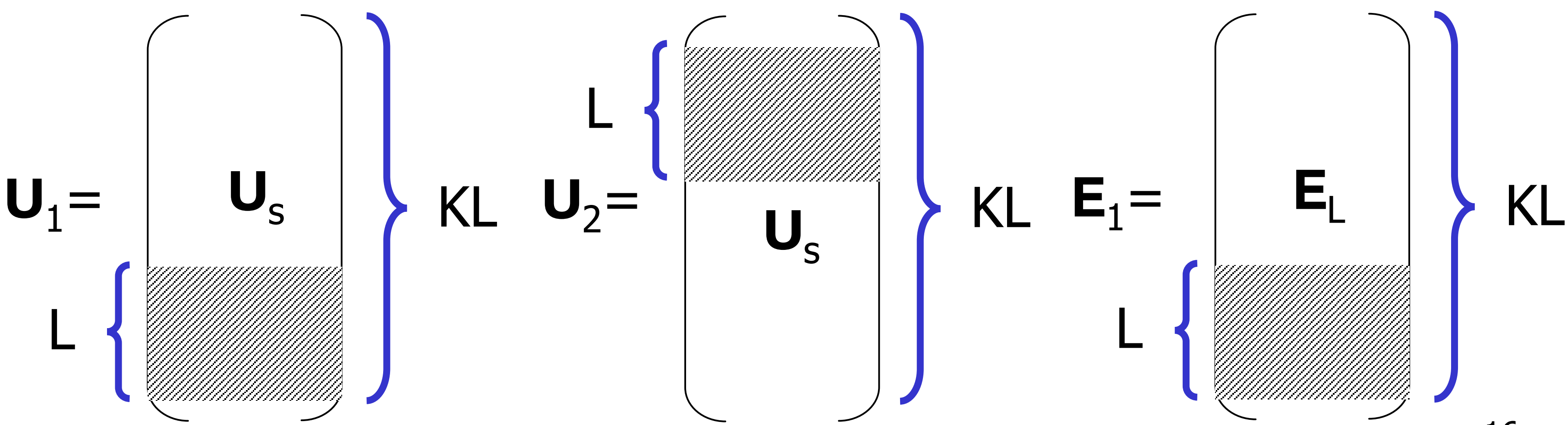
range($\mathbf{E}_L$) = range($\mathbf{U}_s$) = N_s

T: unique N_s x N_s nonsingular matrix

$$\mathbf{U}_1 \equiv \mathbf{E}_1 \mathbf{T}$$

$$\mathbf{U}_2 \equiv \mathbf{E}_1 \mathbf{Y}_d \mathbf{T}$$

$\mathbf{Y}_d = \text{diag}\left(y_1, y_2, \ldots, y_{N_s}\right) \Longleftrightarrow f_{1k}$



Matrix Pencil and Extracting f_1

$$\mathbf{U}_1 = \mathbf{E}_1 \mathbf{T}$$

$$\mathbf{U}_2 = \mathbf{E}_1 \mathbf{Y}_d \mathbf{T}$$

$\mathbf{Y}_d = \mathrm{diag}\left(y_1, y_2, \ldots, y_{N_s}\right) \Leftrightarrow f_{1k}$

$$\mathbf{U}_2 - \lambda \mathbf{U}_1 = \mathbf{E}_1 \left(\mathbf{Y}_d - \lambda \mathbf{I}\right) \mathbf{T} = 0$$

Rank reducing number is y_k, $k=0,1,\ldots,N_s$

Solve for generalized eigenvalue of $\mathbf{U}_2-\lambda\mathbf{U}_1$ to obtain f_{1k}

Matrix Pencil for f_2

$$\mathbf{U}_{sP} \equiv \mathbf{P}\mathbf{U}_s$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}^T(1) \\ \mathbf{p}^T(1+L) \\ \vdots \\ \mathbf{p}^T(1+(K-1)L) \\ \mathbf{p}^T(2) \\ \mathbf{p}^T(2+L) \\ \vdots \\ \mathbf{p}^T(2+(K-1)L) \\ \vdots \\ \mathbf{p}^T(L) \\ \mathbf{p}^T(L+L) \\ \vdots \\ \mathbf{p}^T(L+(K-1)L) \end{bmatrix}$$

Recall

$$\mathbf{E}_L \equiv \begin{bmatrix} \mathbf{Z}_L \\ \mathbf{Z}_L\mathbf{Y}_d \\ \vdots \\ \mathbf{Z}_L\mathbf{Y}_d^{K-1} \end{bmatrix} \Rightarrow \mathbf{E}_{LP} \equiv \mathbf{P}\mathbf{E}_L = \begin{bmatrix} \mathbf{Y}_L \\ \mathbf{Y}_L\mathbf{Z}_d \\ \vdots \\ \mathbf{Y}_L\mathbf{Z}_d^{K-1} \end{bmatrix}$$

$$\mathbf{Y}_L \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & \cdots & y_{N_s} \\ & & & \\ y_1^{K-1} & y_2^{K-1} & \cdots & y_{N_s}^{K-1} \end{bmatrix}$$

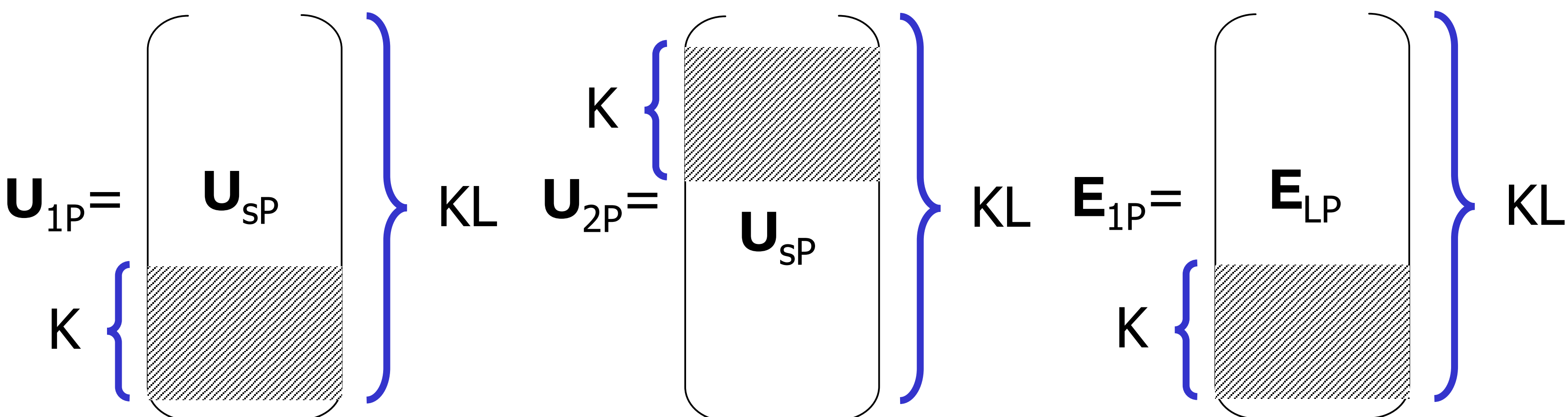
$$\mathbf{Z}_d = \mathrm{diag}(z_1, z_2, \ldots, z_{N_s}) \Leftrightarrow f_{2k}$$

Matrix Pencil for f_2

$$\mathbf{U}_{1P} \equiv \mathbf{E}_{1P}\mathbf{T}$$

$$\mathbf{U}_{2P} \equiv \mathbf{E}_{1P}\mathbf{Z}_d\mathbf{T}$$

T: unique $N_s \times N_s$ nonsingular matrix



Matrix Pencil and Extracting f_2

$$\mathbf{U}_{1P} \equiv \mathbf{E}_{1P}\mathbf{T}$$

$$\mathbf{U}_{2P} \equiv \mathbf{E}_{1P}\mathbf{Z}_d\mathbf{T}$$

$\mathbf{Z}_d = \mathrm{diag}(z_1, z_2, \ldots, z_{N_s}) \Leftrightarrow f_{2k}$

$$\mathbf{U}_{2P} - \lambda\mathbf{U}_{1P} = \mathbf{E}_{1P}(\mathbf{Z}_d - \lambda\mathbf{I})\mathbf{T} = 0$$

Rank reducing number is z_k, $k=0,1,\ldots,N_s$

Solve for generalized eigenvalue of $\mathbf{U}_{2P}$-λ$\mathbf{U}_{1P}$ to obtain f_{2k}

MEMP: Pairing

$$J_s(k,l) = \sum_{t=1}^{N_s} \left\| \mathbf{u}_t^H \mathbf{e}_L(y_k, z_l) \right\|$$

$\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_{\mathbf{N}_s}$ ⟺ Signal subspace

$$\mathbf{E}_L \equiv \begin{bmatrix} \mathbf{Z}_L \\ \mathbf{Z}_L \mathbf{Y}_d \\ \vdots \\ \mathbf{Z}_L \mathbf{Y}_d^{K-1} \end{bmatrix}$$

Correlating signal subspace with known frequency pairs

MMEMP – Simultaneous Frequency Estimation

Recall $\mathbf{U}_2 - \lambda\mathbf{U}_1 = \mathbf{E}_1\left(\mathbf{Y}_d - \lambda\mathbf{I}\right)\mathbf{T}$

Rank reducing number is y_k, $k=0,1,\ldots,N_s$

$$\mathbf{U}_2\mathbf{w}_k = \lambda\mathbf{U}_1\mathbf{w}_k$$

$$\mathbf{U}_2\mathbf{W} = \mathbf{U}_1\mathbf{W}\mathbf{Y}_d$$

$$\left(\mathbf{U}_1^H\mathbf{U}_1\right)^{-1}\mathbf{U}_1^H\mathbf{U}_2\mathbf{W} = \mathbf{U}_1^+\mathbf{U}_2\mathbf{W} = \mathbf{W}\mathbf{Y}_d$$

$$\mathbf{Y}_d = \mathbf{W}^{-1}\mathbf{U}_1^+\mathbf{U}_2\mathbf{W}$$

MMEMP – Simultaneous Frequency Estimation

Recall $\mathbf{U}_2 - \lambda\mathbf{U}_1 = \mathbf{E}_1(\mathbf{Y}_d - \lambda\mathbf{I})\mathbf{T}$

U_2-λU_1 and **E_1(Y_d- λI)T** shares same eigenvalue and eigenvector

$$\mathbf{U}_1^{+}\mathbf{U}_2 = \mathbf{W}\mathbf{Y}_d\mathbf{W}^{-1}$$

$\mathbf{E}_1\mathbf{Y}_d\mathbf{T}\mathbf{w}_k = \lambda\mathbf{E}_1\mathbf{T}\mathbf{w}_k$ and $\lambda \Rightarrow \mathbf{Y}_d$

$$\mathbf{E}_1\mathbf{Y}_d\mathbf{T}\mathbf{W} = \mathbf{E}_1\mathbf{T}\mathbf{W}\mathbf{Y}_d$$

$$\mathbf{Y}_d\mathbf{T}\mathbf{W} = \mathbf{T}\mathbf{W}\mathbf{Y}_d$$

$$\mathbf{T}^{-1}\mathbf{Y}_d\mathbf{T} = \mathbf{W}\mathbf{Y}_d\mathbf{W}^{-1}$$

$$\mathbf{W} = \mathbf{T}^{-1}$$

MMEMP – Simultaneous Frequency Estimation

Given $\mathbf{W} = \mathbf{T}^{-1}$

$$\mathbf{U}_{2P} - \lambda \mathbf{U}_{1P} = \mathbf{E}_{1P}\left(\mathbf{Z}_d - \lambda \mathbf{I}\right)\mathbf{T}$$

$$\mathbf{U}_{1P}^{+}\mathbf{U}_{2P} = \mathbf{R}\mathbf{Z}_d\mathbf{R}^{-1} = \mathbf{T}^{-1}\mathbf{Z}_d\mathbf{T} = \mathbf{W}\mathbf{Z}_d\mathbf{W}^{-1}$$

$$\mathbf{U}_{1P}^{+}\mathbf{U}_{2P} = \mathbf{W}\mathbf{Z}_d\mathbf{W}^{-1}$$

$$\mathbf{Z}_d = \mathbf{W}^{-1}\mathbf{U}_{1P}^{+}\mathbf{U}_{2P}\mathbf{W}$$

Simultaneous Estimation

$$f_{1k}: \quad \mathbf{Y}_d = \mathbf{W}^{-1}\mathbf{U}_1^{+}\mathbf{U}_2\mathbf{W}$$

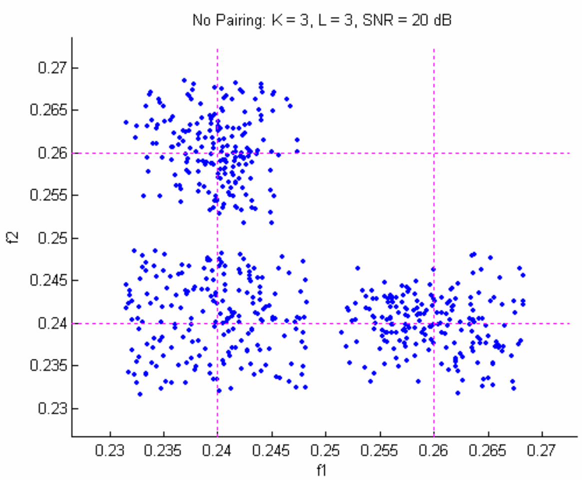
$$f_{2k}: \quad \mathbf{Z}_d = \mathbf{W}^{-1}\mathbf{U}_{1P}^{+}\mathbf{U}_{2P}\mathbf{W}$$

$$\mathbf{Y}_d = \mathrm{diag}\left(y_1, y_2, \ldots, y_{N_s}\right) \Leftrightarrow f_{1k}$$

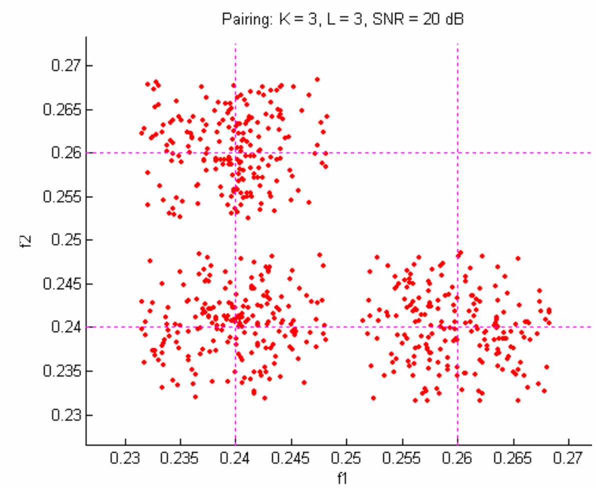
$$\mathbf{Z}_d = \mathrm{diag}\left(z_1, z_2, \ldots, z_{N_s}\right) \Leftrightarrow f_{2k}$$

Results

MMEMP

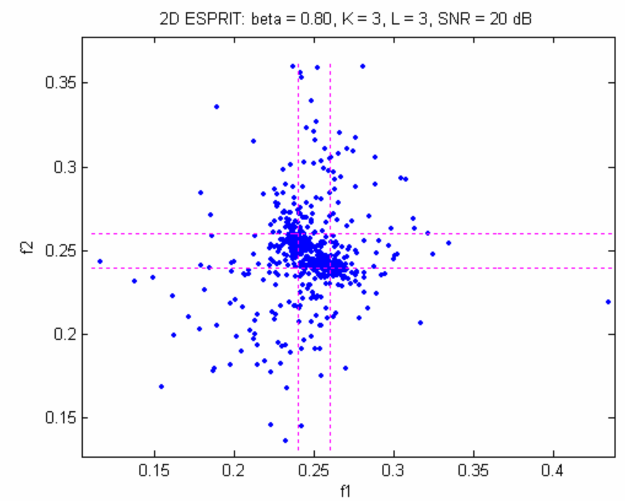


MEMP



2-D ESPRIT

β = 0.8



(f1,f2) = (0.26,0.24)

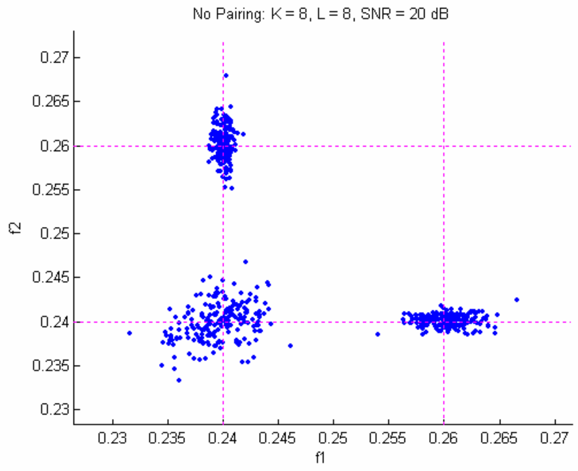
(f1,f2) = (0.24,0.24)

(f1,f2) = (0.24,0.26)

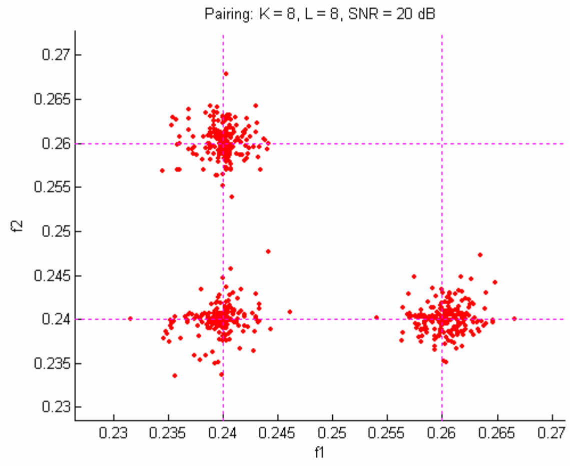
200 independent estimates, M=N=20, K=L=3, SNR = 20 dB

Results

MMEMP

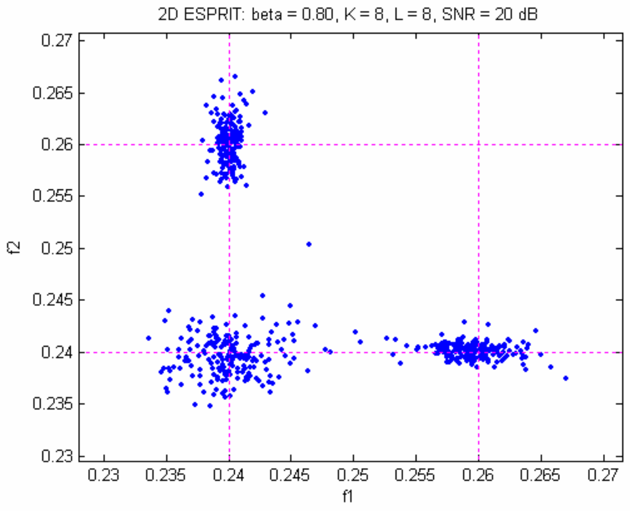


MEMP



2-D ESPRIT

β = 0.8



(f1,f2) = (0.26,0.24)

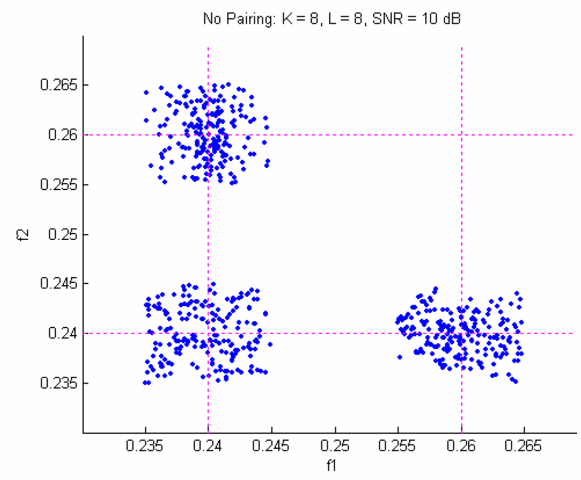
(f1,f2) = (0.24,0.24)

(f1,f2) = (0.24,0.26)

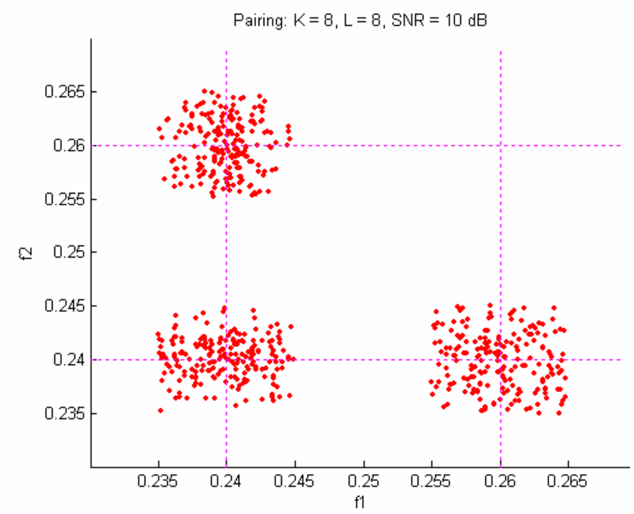
200 independent estimates, M=N=20, K=L=8, SNR = 20 dB

Results (Low SNR)

MMEMP

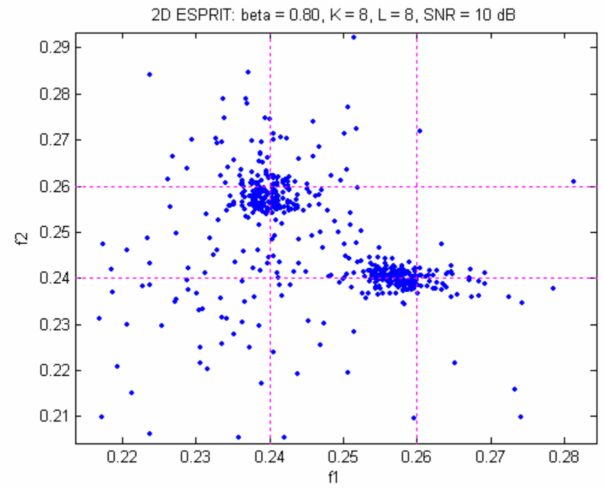


MEMP



2-D ESPRIT

β = 0.8



(f1,f2) = (0.26,0.24)

(f1,f2) = (0.24,0.24)

(f1,f2) = (0.24,0.26)

200 independent estimates, M=N=20, K=L=8, SNR = 10 dB

Results

MMEMP: 200 independent estimates, K=L=6, SNR=20 dB

f_1	Bias ($\times 10^{-4}$)	Dev ($\times 10^{-3}$)	CRB ($\times 10^{-3}$)	f_2	Bias ($\times 10^{-4}$)	Dev ($\times 10^{-3}$)	CRB ($\times 10^{-3}$)
0.26	2.19	**2.22**	0.40	0.24	0.84	**0.82**	0.32
0.24	-5.48	**3.17**	0.31	0.24	-1.80	**2.78**	0.31
0.24	0.66	**0.74**	0.32	0.26	2.54	**2.42**	0.40

Results

MEMP: 200 independent estimates, K=L=6, SNR=20 dB

f_1	Bias ($\times 10^{-4}$)	Dev ($\times 10^{-3}$)	CRB ($\times 10^{-3}$)	f_2	Bias ($\times 10^{-4}$)	Dev ($\times 10^{-3}$)	CRB ($\times 10^{-3}$)
0.26	2.19	**2.22**	0.40	0.24	-1.35	**2.14**	0.32
0.24	-1.36	**2.05**	0.31	0.24	0.11	**2.19**	0.31
0.24	-3.46	**2.57**	0.32	0.26	2.82	**2.43**	0.40

Complexity

Number of Multiplications, *$K>N_s>>1$, $L>N_s>>1$, $M>>1$, $N>>1$*

	SVD of $\mathbf{X}_e$	$\mathbf{U}_1^+\mathbf{U}_2$ or $\mathbf{U}_{1P}^+\mathbf{U}_{2P}$	Pairing
MMEMP	$2KL\left(M-\frac{K}{2}\right)\left(N-\frac{L}{2}\right)+5K^3L$	$3N_s^2KL+7N_s^3$	N/A
MEMP	$2KL\left(M-\frac{K}{2}\right)\left(N-\frac{L}{2}\right)+5K^3L$	$3N_s^2KL+10N_s^3$	$\frac{1}{2}N_s^3KL$
2-D ESPRIT	$2KL\left(M-\frac{K}{2}\right)\left(N-\frac{L}{2}\right)+5K^3L$	$3N_s^2KL+9N_s^3$	N/A

Conclusion

- Proposed MMEMP
  - Estimate f_1 and f_2 simultaneously
  - Less computational complexity than MEMP and 2-D ESPRIT
  - More accurate and consistent pairing than MEMP
- MEMP
  - Estimate f_1 and f_2 separately
  - Requires Pairing
    - Not always accurate
  - Higher computational complexity than MMEMP and 2-D ESPRIT
- 2-D ESPRIT
  - Estimate f_1 and f_2 simultaneously
  - Higher computational complexity than MMEMP
  - Does not perform as well as the MMEMP method in low SNR

References

- Y. Hua, "Estimating Two-Dimensional Frequencies by Matrix Enhancement and Matrix Pencil", *IEEE Trans. on Signal Processing*, vol. 49(9), p. 2267-2280, Sep. 1992.
- S. Rouquette and M. Najim, "Estimation of Frequencies and Damping Factors by Two-Dimensional ESPRIT Type Methods", *IEEE Trans. On Signal Processing*, vol. 49(1), pp. 237-245, Jan. 2001.
- J.H. McClellan, "Multidimensional Spectral Estimation", *Proc. of the IEEE*, vol. 70(9), Sep. 1982.
- H.M. Ibrahim and R.R. Gharieb, "Estimating Two-Dimensional Frequencies by a Cumulant-Based FBLP Method", *IEEE Trans. on Signal Processing*, vol. 47(1), p. 262-266, Jan. 1999.

Other Research: Parametric System Estimation

Blind FIR Channel Estimation/Equalization and Tracking

- 2^{nd}-order statistics based minimal transmit redundancy space-time FIR precoder-blind equalizer
- 2^{nd}-order statistics based iterative subspace tracking for FIR SIMO channels

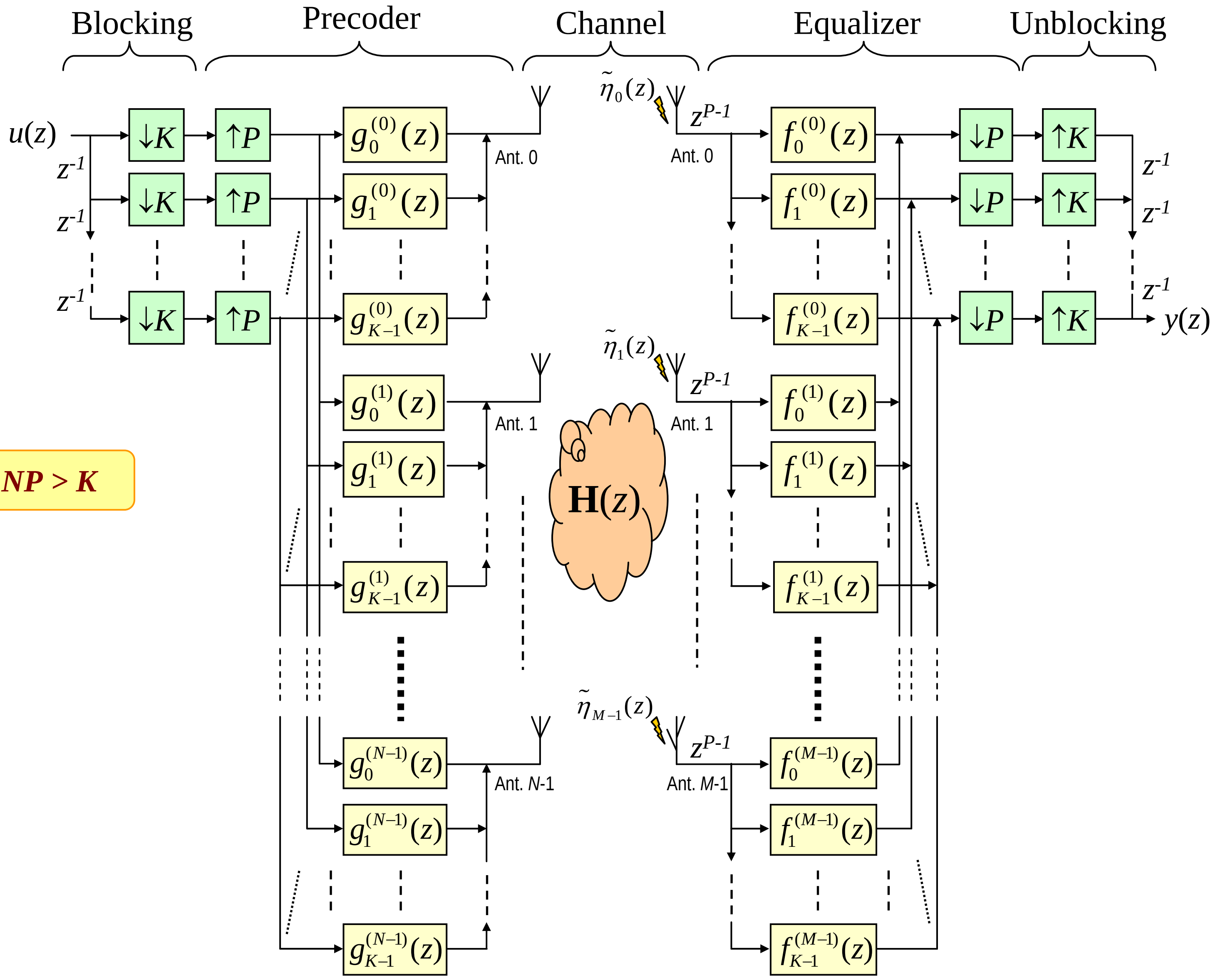
- 4^{th}-order statistics based FIR SISO blind channel estimator
- 4^{th}-order statistics based minimal transmit redundancy space-time FIR precoder-blind equalizer
- 4^{th}-order statistics based iterative space-time FIR precoder-blind equalizer

Minimal Transmit Redandancy Space-Time Precoder-Equalizer System

- Still able to equalize the channel when transmit redundancy is less than the channel order
- Generalizes ALL block based linear precoder-equalizer systems, e.g. OFDM

X.-G. Xia, G. Wang and P. Fan, "Space-Time Modulated Codes for Memory Channels: Capacity and Information Rates, Zero-Forcing Decision Feedback Equalizer", *Proc. of the IEEE Sensor Array and Multichannel Signal Processing Workshop*, pp. 183-187, Mar. 2000.

Space-Time Precoder-Equalizer Structure Using Nonmaximally Decimated Filter Bank



$NP > K$

SOS Based MTR Space-Time FIR Precoder-Blind Equalizer

- Requires ONLY [Hua & Tugnait 2000]
  - $\boldsymbol{H}(z)$ irreducible
  - $\boldsymbol{S}_{uu}(z)$ diagonal with distinct diagonal elements
  - Less restrictive than previously requiring $\boldsymbol{H}(z)$ to be irreducible and column reduced
- Design FIR MIMO blind channel zero-forcing equalizer
  - $\mathbf{F}(z)\mathcal{G}(z) = \Sigma\mathbf{P}\Lambda(z)$ → scalar, permutation, and delay ambiguities
    - Σ is a scalar diagonal matrix
    - $\mathbf{P}$ is a permutation matrix (with one 1 on each row/column)
    - $\Lambda(z)$ is a polynomial diagonal matrix with monic polynomial as diagonal elements
  - Without loss of generality, $\mathbf{F}(z)\mathcal{G}(z) = z^{-r}\mathbf{I}_{KxK}$

HOS Based MTR Space-Time FIR Precoder-Blind Equalizer

- Extend to FIR MIMO channel equalizer design
- HOS – robust towards Gaussian noise
- Design FIR MIMO blind channel zero-forcing equalizer
  - Without loss of generality, $\mathbf{F}(z)\mathcal{G}(z) = z^{-r}\mathbf{I}_{KxK}$
- Take advantage of Minimal Transmit Redundancy property of precoder-equalizer structure

Iterative Subspace Estimation and Tracking for FIR SIMO Channels

- Reduced complexity channel tracking using subspace method
  - Constrained gradient-based adaptive subspace tracking algorithm
  - $O(n^2)$ vs. $O(n^3)$

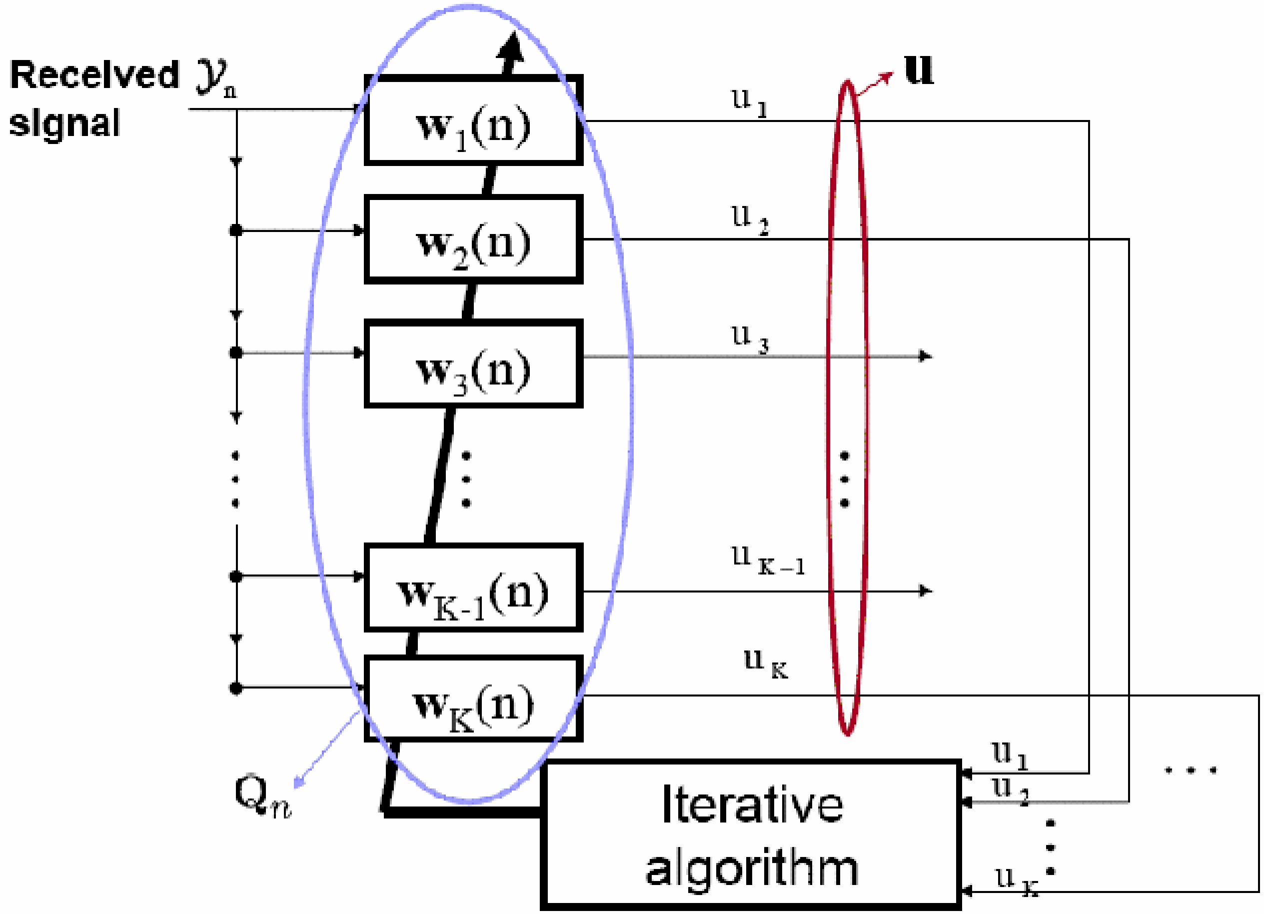


Figure 4.2: Adaptive subspace filter bank for tracking eigenvectors.